New quasilocal conserved operators in XXX spin 1/2 chain

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Quasilocal conserved operators in isotropic Heisenberg spin 1/2 chain

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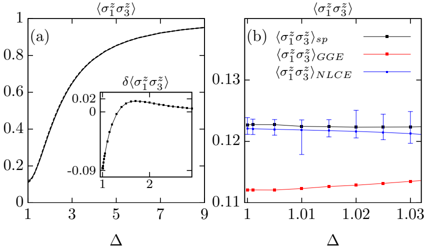
Composing higher auxiliary–spin transfer matrices and their derivatives, we construct a family of quasilocal conserved operators of isotropic Heisenberg spin 1/2 chain and rigorously establish their linear independence from the well-known set of local conserved charges.

The problem of *MISSING QUASILOCAL CONSERVED CHARGE(s)*

Generalized Gibbs ensemble $\rho_{\rm GGE} = \exp(-\sum_{j=1}^{\infty} \beta_j Q_j)$ for the steady state after a quantum quench of XXZ Hamiltonian gives **incorrect results**!

$$H = \sum_{x=1}^{n-1} (2\sigma_x^+ \sigma_{x+1}^- + 2\sigma_x^- \sigma_{x+1}^+ + \Delta \sigma_x^{\rm z} \sigma_{x+1}^{\rm z})$$

B. Wouters *et al.*, Phys. Rev. Lett. **113**, 117202 (2014); M. Brockmann *et al.*, J. Stat. Mech. P12009 **(2014)**:



B. Pozsgay *et al.*, Phys. Rev. Lett. **113**, 117203 (2014); M. Mestyan *et al.*, J. Stat. Mech. P04001 **(2015)**:

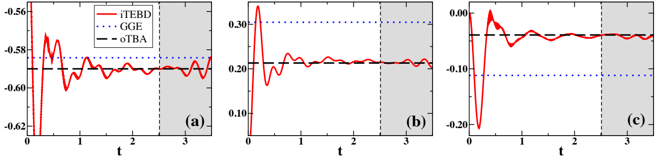


FIG. 1: Numerical simulation of the time evolution of correlation functions (a) $\langle\sigma_1^z\sigma_2^z\rangle$, (b) $\langle\sigma_1^z\sigma_3^z\rangle$, (c) $\langle\sigma_1^z\sigma_4^z\rangle$

Conserved charges and bounds on (generalized) Drude weights

$$D_A = \lim_{t\to\infty} \lim_{n\to\infty} \frac{\beta}{n\tau} \int_0^\tau \mathrm{d}t \langle A(t) A(0) \rangle_\beta$$

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$$D_A = (A, \bar{A}) = (\bar{A}, \bar{A}), \quad \bar{A} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t e^{\mathrm{i}Ht} A e^{-\mathrm{i}Ht}, \quad (A, B) = \langle A^\dagger B\rangle_\beta.$$

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For integrable systems, Zotos et. al (1997) suggested to use Mazur/Suzuki (1969/1971) bound, estimating Drude weight in terms of local (or quasi-local!) conserved operators Q_j, $[H, Q_j] = 0$:

$$D_A \geq \lim_{n\to\infty} \frac{\beta}{2n} \sum_m \frac{|(A, Q_m)|^2}{(Q_m, Q_m)}$$

where operators Q_m are chosen mutually orthogonal $(Q_m, Q_k) = 0$ for $m \neq k$.

Local conserved charges of a (periodic) chain on n sites are translationally invariant sums of local operators q_k supported on k sites

$$Q_k = \sum_{x=0}^{n-1} \hat{S}^x (q_k \otimes \mathbb{1}_{2^{n-k}}).$$

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Local operators Q – considered as series of operators for increasing size n – obey the following properties:

1. The Hilbert-Schmidt norm is linearly extensive

$$\|Q\|^2_{\mathrm{HS}} = (Q, Q) \propto n.$$

2. For any locally supported $a = a_k \otimes \mathbb{1}_{2^{n-k}}$, (a, Q) is independent of n.

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Definition (*quasilocality*)

A non-local translationally invariant operator Q, $Q = \hat{\mathcal{S}}(Q)$ (again, considered as a series w.r.t. a sequence of sizes n) is quasi-local if it satisfies (1) and (2).

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The effect of quasilocal conserved operators to statistical mechanics is arguably as important as that of local operators.

Numerical search for a complete set of quasilocal charges

M. Mierzejewski, P. Prelovšek, T. P, Phys. Rev. Lett. **114**, 140601 (2015)

Observation: If A general local translationally invariant operator, then $\bar{A}$ is quasilocal and conserved.

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Take a maximal support M and consider all $\mathcal{N}_M \sim 4^M$ local TI operators $O_{\underline{s}} = \sum_x \hat{\mathcal{S}}^x(\sigma_1^{s_1} \ldots \sigma_M^{s_M})$, and define a time averaging matrix

$$K_{\underline{s},\underline{s}'} = (\bar{O}_{\underline{s}} | \bar{O}_{\underline{s}'}) = (O_{\underline{s}} | \bar{O}_{\underline{s}'}).$$

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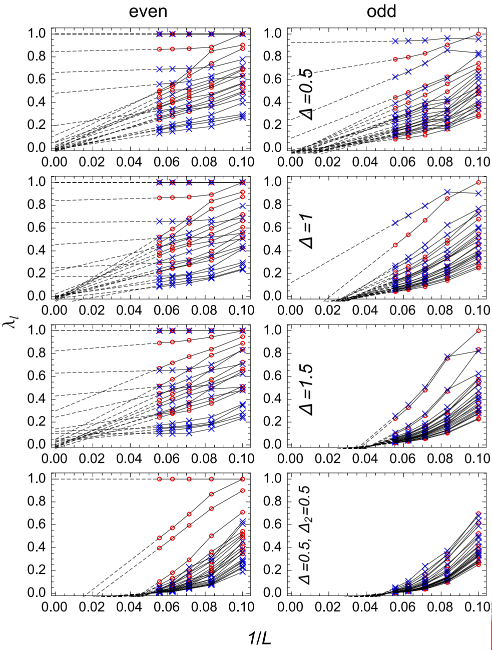
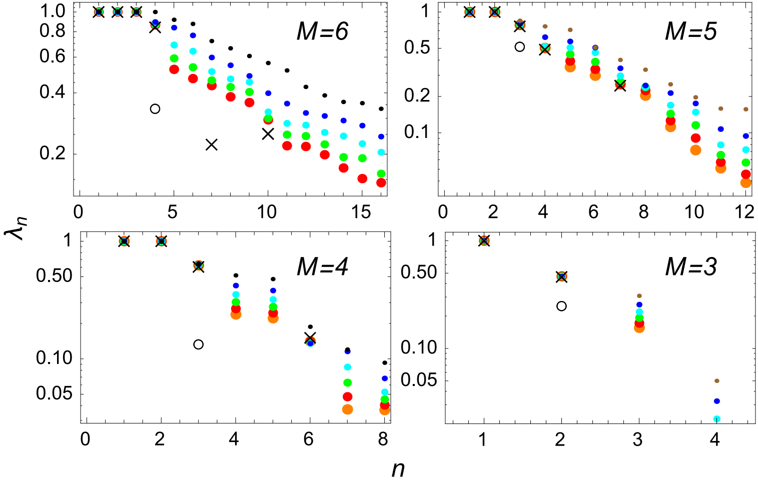
An effective rank of the matrix K gives an effective number of independent quasi-local conserved quantities.
More precisely: eigenvectors of eigenvalue 1 correspond to local conserved charges, and dominating subunital eigenvalues $\lambda_l < 1$ correspond to *principal quasilocal conserved charges*

THE RESULTS (here $L \equiv n$):

The size $(1/L)$ scaling of leading eigenvalues λ_n of matrix K with $M = 6$, corresponding to symmetric (red) or antisymmetric (blue) eigenoperators with respect to time reversal. Left/right column shows even/odd parity sectors, while rows indicate different regimes of integrabke (upper three rows) and non-integrable (lower row) with parameters indicated in the panel. Dashed lines indicate $1/L$ extrapolation to TL which in some cases provide clear indication of existence of QLCQ $\lambda_n|_{L\to\infty} > 0$, beyond the local eigenoperators with $\lambda_n = 1$.





Dependence of leading eigenvalues λ_n of K for even parity, symmetric time reversal (E-R) sector if eigenoperators in isotropic HM $\Delta = 1$. Different panels indicate decreasing support sizes $M = 6, 5, 4, 3$, while decreasing sizes of points and colors indicate the system size $L = 20$ (orange), 18 (red), 16 (green), 14 (cyan), 12 (blue), 10 (black). The extrapolated $L \to \infty$ values are indicated with crosses when in the range of the plot.

But that's all numerics.

Can we do better than that and construct new quasilocal conserved charges analytically?

For simplicity we from now on consider only the isotropic Heisenberg (XXX) chain with $\Delta = 1$

Consider $2s+1$ dimensional spin$-s$ auxiliary space $\mathcal{H}_{\mathrm{a}} = \mathcal{V}_s$ with SU(2) generators represented as

$$\mathbf{s}^{\mathrm{z}}|m\rangle = m|m\rangle,\ \mathbf{s}^{\pm}|m\rangle = \sqrt{(s+1\pm m)(s\mp m)}|m\pm 1\rangle$$

and define Lax operators acting over $\mathcal{H}_{\mathrm{p}} \otimes \mathcal{H}_{\mathrm{a}}$, $\mathcal{H}_{\mathrm{p}} = \mathcal{V}_{1/2}^{\otimes n}$,

$$\mathbf{L}_{x,\mathrm{a}}(\lambda) = \lambda \mathbb{1} + \sigma_x^{\mathrm{z}} \mathbf{s}_{\mathrm{a}}^{\mathrm{z}} + \sigma_x^{+} \mathbf{s}_{\mathrm{a}}^{-} + \sigma_x^{-} \mathbf{s}_{\mathrm{a}}^{+} = \lambda \mathbb{1} + \vec{\sigma}_x \cdot \vec{\mathbf{s}}_{\mathrm{a}},$$

in turn defining a commuting set of transfer matrices

$$T_s(\lambda) = \mathrm{tr}_{\mathrm{a}} \mathbf{L}_{0,\mathrm{a}}(\lambda) \mathbf{L}_{1,\mathrm{a}}(\lambda) \cdots \mathbf{L}_{n-1,\mathrm{a}}(\lambda),$$
$$[T_s(\lambda), T_{s'}(\lambda')] = 0, \quad \forall s, s', \lambda, \lambda'.$$

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The fundamenal TM $T_{1/2}(\lambda)$ is generating all local Hermitian conserved charges

$$Q_k = -{\rm i}\partial_t^{k-1}\log T_{1/2}(\tfrac{1}{2}+{\rm i}t)|_{t=0} = \sum_{x=0}^{n-1}\hat{\mathcal{S}}^x\left(q_k\otimes\mathbb{1}_{2^{n-k}}\right),$$

with $H_{XXX} = Q_2$.

Theorem (arXiv:1506.05049):

Traceless operators $X_s(t)$, $s \in \frac{1}{2}\mathbb{Z}^+, t \in \mathbb{R}$, defined as

$$
\begin{aligned}
X_s(t) &= [\tau_s(t)]^{-n} \left\{ T_s(-\tfrac{1}{2} + \mathrm{i}t) T_s'(\tfrac{1}{2} + \mathrm{i}t) \right\}, \\
\tau_s(t) &= -t^2 - \left(s + \tfrac{1}{2}\right)^2,
\end{aligned}
$$

where $T_s'(\lambda) \equiv \partial_\lambda T_s(\lambda)$ and $\{A\} \equiv A - (\mathrm{tr}\, A)\mathbb{1}/(\mathrm{tr}\, \mathbb{1})$, are quasilocal for all s, t and linearly independent from $\{Q_k; k \geq 2\}$ for $s > \frac{1}{2}$.

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Inspiration: for $s = 1/2$, TM is asymptitically, $n \to \infty$, a unitary operator

$$T_{1/2}(\tfrac{1}{2} + \mathrm{i}t) \simeq \exp\left(\mathrm{i} \sum_{k=1}^{\infty} Q_{k+1} t^k / k! \right),$$

(`Fagotti and Essler, JSTAT P07012 (2013)`) hence $X_{1/2}(t)$ is a logarithmic derivative, since $T_s^\dagger(\lambda) \equiv T_s^T(\bar{\lambda}) = (-1)^n T_s(-\bar{\lambda})$.

MPO form of a product of a pair of TMs, and a trace of a quadruple of TMs

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\begin{aligned}
T_s(\mu)T_s(\lambda) &= \mathrm{tr}_{a_1,a_2} \prod_{x=0}^{n-1} \left(\sum_\alpha \mathbb{L}_s^\alpha(\mu,\lambda)\sigma_x^\alpha \right), \\
\mathbb{L}_s^0(\mu,\lambda) &= \lambda\mu\mathbb{1} + \vec{\mathbf{s}}_{a_1}\cdot\vec{\mathbf{s}}_{a_2}, \\
\vec{\mathbb{L}}_s(\mu,\lambda) &= \mathrm{i}\,\vec{\mathbf{s}}_{a_1}\times\vec{\mathbf{s}}_{a_2} + \lambda\vec{\mathbf{s}}_{a_1} + \mu\vec{\mathbf{s}}_{a_2}, \\
(T_s(\mu)T_s(\lambda), T_{s'}(\mu')T_{s'}(\lambda')) &= \mathrm{tr}_{a_1,a_2,a_3,a_4}\mathbb{T}_{s,s'}^n, \\
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Identity component $\mathbb{L}_s^0 = \mu\lambda\mathbb{1} + \frac{1}{2}(\mathbf{C} - \vec{\mathbf{s}}_{\mathrm{a}_1}^2 - \vec{\mathbf{s}}_{\mathrm{a}_2}^2)$, has the spectrum $\tau_s^j(\mu,\lambda) = \frac{j(j+1)}{2} - s(s+1) + \mu\lambda$, $j = 0, 1, \ldots, 2s$.

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Specializing on one of the two lines

$$
\begin{aligned}
\mathcal{D}^\pm &= \{(\mu_t^\pm,\lambda_t^\pm);\, t\in\mathbb{R}\} \subset \mathbb{C}^2, \\
\mu_t^\pm &:= \mp\tfrac{1}{2} + \mathrm{i}t, \quad \lambda_t^\pm := \pm\tfrac{1}{2} + \mathrm{i}t,
\end{aligned} \tag{1}
$$

the leading eigenvalue of $\mathbb{L}_s^0$ is $\tau_s(t) = -(s+1/2)^2 - t^2$ corresponding to the *singlet* $j=0$ eigenstate (in auxiliary space!)

$$
|\psi_0\rangle = (2s+1)^{-1/2} \sum_{m=-s}^{s} (-1)^{s-m}|m\rangle\otimes|-m\rangle, \tag{2}
$$

Hilbert-Schmidt Kernel

$$
\begin{aligned}
&K_{s,s'}(t,t') := (X_s(t), X_{s'}(t')) = \\
&[\tau_s(t)\tau_{s'}(t')]^{-n}\partial_{\lambda_t^-}\partial_{\lambda_{t'}^+}\Bigg(\mathrm{tr}\,\left[\mathbb{T}_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+)\right]^n \\
&\qquad\qquad - \mathrm{tr}\,\left[\mathbb{L}_s^0(\mu_t^-,\lambda_t^-)\right]^n \mathrm{tr}\,\left[\mathbb{L}_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)\right]^n\Bigg).
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$$-\,\mathrm{tr}\,\left[\mathbb{L}_s^0(\mu_t^-,\lambda_t^-)\right]^n\,\mathrm{tr}\,\left[\mathbb{L}_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)\right]^n\Bigg).$$

Magic key: Precisely on $\mathcal{D}^- \times \mathcal{D}^+$ the leading eigenvalue of $\mathbb{T}_{s,s'}$ factorizes

$$\tau_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+) = \tau_s(t)\tau_{s'}(t')$$

with eigenvector $|\Psi_0\rangle = |\psi_0\rangle \otimes |\psi_0\rangle$.

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with eigenvector $|\Psi_0\rangle = |\psi_0\rangle \otimes |\psi_0\rangle$.

Using Feynman-Hellmann,

$$\partial_{\lambda_{t'}^+}\tau_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+) = \tau_s^0(\mu_t^-,\lambda_t^-)\partial_{\lambda_{t'}^+}\tau_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)$$

one finally obtains the extensivity of HSK

$$K_{s,s'}(t,t') = n[\tau_s(t)\tau_{s'}(t')]^{-1}\partial_{\lambda_t^-}\partial_{\lambda_{t'}^+}\left(\tau_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+)\right.$$
$$\left. -\tau_s^0(\mu_t^-,\lambda_t^-)\tau_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)\right) + \mathcal{O}(e^{-\gamma n}).$$

With some effort, one can obtain an explicit asymptotic form of the HSK

$$K_{s,s'}(t,t') = n\frac{\kappa_{s,s'}(t-t')}{\tau_s(t)\tau_{s'}(t')},$$

$$\kappa_{s,s'}(\tau) = \sum_{l=1}^{2s} \frac{l(l+2(s'-s))(2s+1-l)(2s'+1+l)}{(2s+1)(2s'+1)} c_{s'-s+l}(\tau),$$

$$\text{where} \quad c_s(\tau) := \frac{s}{s^2+\tau^2}.$$

For illustration, we only consider the case $s = 1$, and define

$$\widetilde{X}_1(t) = X_1(t) - \int_{-\infty}^{\infty} \mathrm{d}t' f_t(t') X_{1/2}(t').$$

$f_t(t')$ is determined by minimizing the HS norm $\|\widetilde{X}_1(t)\|^2_{\mathrm{HS}}$, i.e. by the variation

$$\frac{\delta}{\delta f_t(t')} (\widetilde{X}_1(t), \widetilde{X}_1(t)) = 0,$$

resulting in the Fredholm equation of the first kind

$$\int_{-\infty}^{\infty} \mathrm{d}t'' K_{1/2,1/2}(t', t'') f_t(t'') = K_{1/2,1}(t', t),$$

with explicit solution

$$f_t(t') = \frac{8}{9\pi} \frac{1 + {t'}^2}{((3/2)^2 + t^2)((1/2)^2 + (t - t')^2)}.$$

Proof of linear independence from the set of local charges

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with explicit solution

$$f_t(t') = \frac{8}{9\pi} \frac{1 + t'^2}{((3/2)^2 + t^2)((1/2)^2 + (t - t')^2)}.$$

FE also implies that $(\widetilde{X}_1(t), X_{1/2}(t')) \equiv (\widetilde{X}_1(t), Q_k) \equiv 0$, and $\|\widetilde{X}_1(t)\|_{\mathrm{HS}} > 0$.
QED

Numerical example

The simplest linearly independent new quasilocal charge:

$$\widetilde{X}_1(0) = -\frac{7\cdot 2^5}{3^7}\sum_{x=0}^{n-1}\Big(\vec{\sigma}_x\cdot\vec{\sigma}_{x+2} + \frac{155}{252}\vec{\sigma}_x\cdot\vec{\sigma}_{x+3}$$
$$+\frac{16}{63}(\vec{\sigma}_x\cdot\vec{\sigma}_{x+1})(\vec{\sigma}_{x+2}\cdot\vec{\sigma}_{x+3}) - \frac{53}{84}(\vec{\sigma}_x\cdot\vec{\sigma}_{x+2})(\vec{\sigma}_{x+1}\cdot\vec{\sigma}_{x+3})$$
$$-\frac{11}{84}(\vec{\sigma}_x\cdot\vec{\sigma}_{x+3})(\vec{\sigma}_{x+1}\cdot\vec{\sigma}_{x+2})\Big) + \mathrm{h.o.t.}$$

Concluding remarks

- If you don't like continuous families of quasi-local charges, the Taylor coefficients $Q_{s,k}$ should be equally useful

$$X_s(t) = Q_{s,2} + tQ_{s,3} + \frac{t^2}{2} Q_{s,4} \ldots, \quad Q_{s,k+2} = (1/k!) \partial_t^k X_s(t)|_{t=0}.$$

- $Q_{s,k}$ and $X_s(t)$ are Hermitian, for $t \in \mathbb{R}$, so our analysis in fact also proves a general inversion formula, asymptotically as $n \to \infty$

$$T_s^{-1}(\tfrac{1}{2} + \mathrm{i}t) \simeq [\tau_s(t)]^{-1} T_s(-\tfrac{1}{2} + \mathrm{i}t)$$

- This shows that, again asymptotically, $X_s(t)$ are just logarithmic derivatives

$$X_s(t) \simeq -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \log T_s(\tfrac{1}{2} + \mathrm{i}t).$$

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