Eigenstate phase transitions for strong zero modes

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Much Ado About MBL

One fascinating aspect of the recent work on many-body localization is the appearance of eigenstate phase transitions.

These transitions are unconventional (not "thermal") in that they involve only excited states – the ground state is unaffected.

One possible example is that spectral statistics (spacing between energy levels) changes from Poisson to GOE.

Huse, Nandkishore, Oganesyan, Pal and Sondhi

A parafermionic avatar

- Such eigenstate phase transitions typically seem tied up in the physics of disorder.

- Here I give both analytic and numerical evidence for an analogous eigenstate phase transition in $\mathbb{Z}_n$-invariant ``spin''/parafermionic systems.

  Fendley; Jermyn, Mong, Alicea and Fendley; see also review by Alicea and Fendley

- This eigenstate transition does not seem to require disorder.

But first…. Ising/Majorana!

The Hilbert space is a chain of two-state systems $(\mathbb{C}^2)^{\otimes L}$

The Jordan-Wigner transformation defines fermions in terms of strings of spin flips:

$$\psi_{2j-1} = \sigma_j^z \prod_{k=1}^{j-1} \sigma_k^x \qquad\qquad \psi_{2j} = i\sigma_j^x \psi_{2j-1}$$

String flips all spins behind site j

$$\{\psi_a, \psi_b\} = 2\delta_{ab}$$

The 1d Ising Hamiltonian is bilinear in fermions:

$$H = -\sum_{j=1}^{L} u_{2j-1}\,\sigma_j^x - \sum_{j=1}^{L-1} u_{2j}\,\sigma_j^z \sigma_{j+1}^z$$

$$= i \sum_{a=1}^{2L-1} u_a \psi_a \psi_{a+1}$$

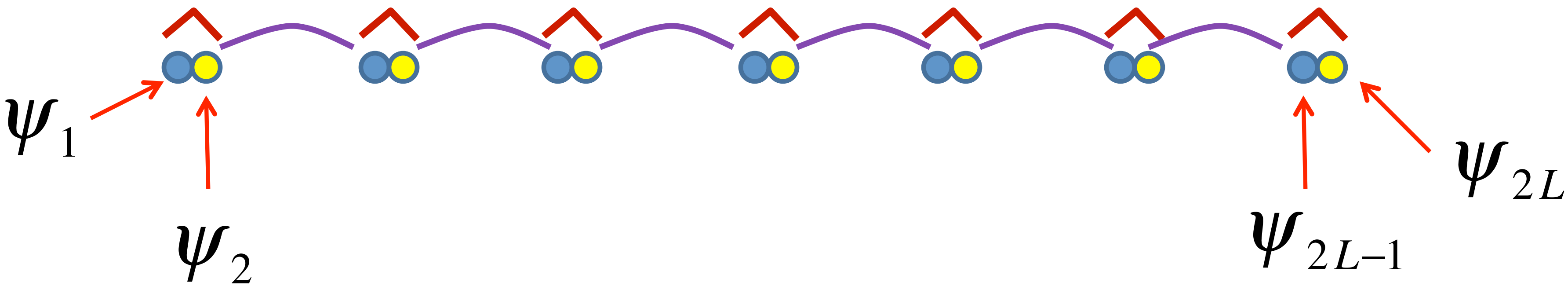
These are open boundary conditions and arbitrary couplings u_a .

$\mathbb{Z}_2$ symmetry operator flips all spins:

$$(-1)^F = \prod_{j=1}^{L} \sigma_j^x = (-1)^L \prod_{a=1}^{2L} \psi_a$$

The Hamiltonian in terms of fermions

A pictorial representation for free boundary conditions:



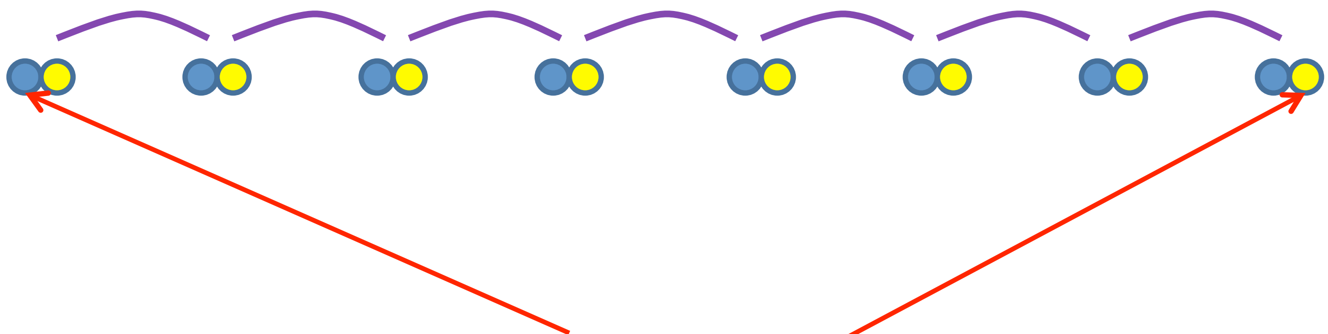
$$H = -\sum_{j=1}^{L} u_{2j-1}\,\sigma_j^x - \sum_{j=1}^{L-1} u_{2j}\,\sigma_j^z \sigma_{j+1}^z$$

$$= i\sum_{a=1}^{2L-1} u_a \psi_a \psi_{a+1}$$

Extreme limits:

- u_{2j} = 0 (disordered in spin language):

- $u_{2j\text{-}1}$ = 0 (ordered in spin language):



The fermions on the edges, ψ_1 and ψ_{2L}, do not appear in H when u_1= 0. They commute with H!

Strong zero modes

- When $f = 0$, the operators ψ_1 and ψ_{2L} commute with H but anticommute with $(-1)^F$.

- They are exact edge zero modes – they map each state with $(-1)^F = \pm 1$ to a state with $(-1)^F = \mp 1$ having the same energy.

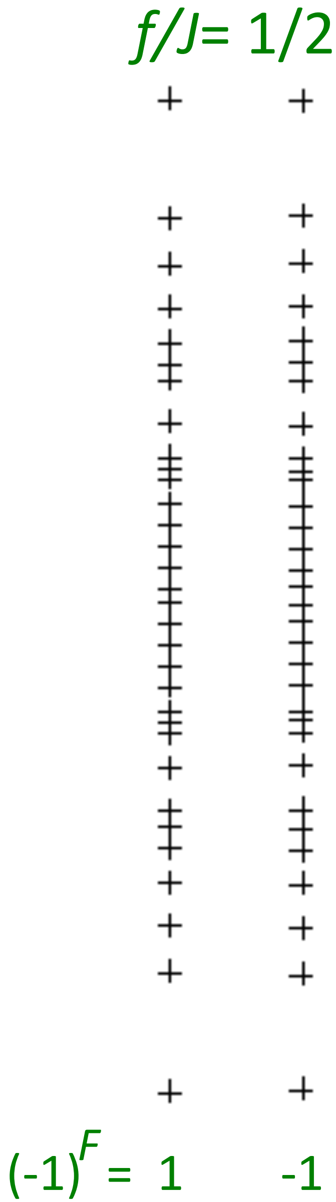
- We call this these strong zero modes. It means the spectrum in different symmetry sectors is identical.

- A strong zero mode is not necessary for topological order; only a weak zero mode guaranteeing ground-state degeneracy is.

- The strong zero mode however is likely more useful for quantum computation.

Huse et al; Bauer and Nayak; Bahri, Vosk, Altman and Vishwanath

Do the strong zero modes persist away from the trivial limit?



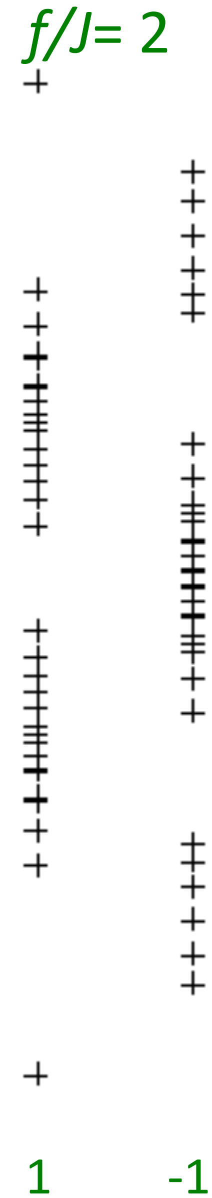
Take couplings uniform in space.

Flip term: $f = u_{2j-1}$

Potential: $J = u_{2j}$

Trivial limit is $f=0$.

$L=6$



The exact strong zero mode

- The strong edge zero modes persist for all $f < J$: the series

$$\Psi = \chi_1 + \frac{f}{J}\chi_3 + \left(\frac{f}{J}\right)^2 \chi_5 + \dots$$

commutes with H up to exponentially small terms:

$$[H, \Psi] \sim \left(\frac{f}{J}\right)^L$$

Kitaev

- When $f < J$, Ψ is localized near the edge, and normalizable: $\Psi^2 = \frac{1}{1-(f/J)^2}$

- This still works when the couplings are ``disordered'', i.e. when f and J vary in space:

$$\Psi = \chi_1 + \frac{u_1}{u_2}\chi_3 + \frac{u_1}{u_2}\frac{u_3}{u_4}\chi_5 + \dots$$

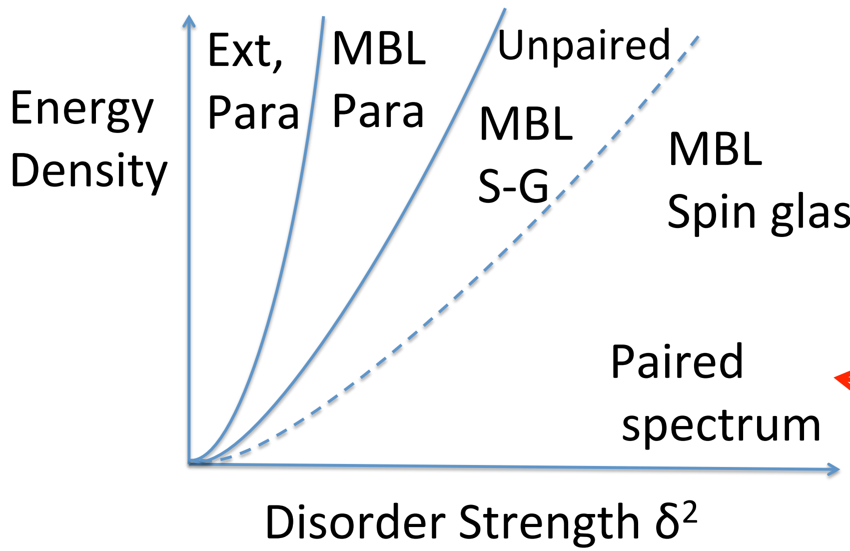
Ising spin order corresponds to topological order!

It is robust against disorder, as one expects in a topological phase.

Does the strong zero mode survive interactions?

- There seems to be a prejudice against in the literature against it, since in Ising a four-fermi term breaks integrability.

- It is argued to occur when disorder is strong enough:



Huse, Nandkishore, Oganesyan, Pal and Sondhi

Their name for the strong zero mode phase

- Numerical work indicates it does indeed occur with strong disorder.

Bauer and Nayak; Bahri, Vosk, Altman and Vishwanath

But why is disorder necessary?

- There is strong evidence that strong zero modes occur without disorder in the parafermion case when the interactions are chiral.

  Fendley; Jermyn, Mong, Alicea and Fendley

- There is an argument, supported by numerical calculations, that they survives in the Ising case.

  Kells

- In Ising the strong zero mode exists throughout the ordered phase; presumably interactions do not change this.

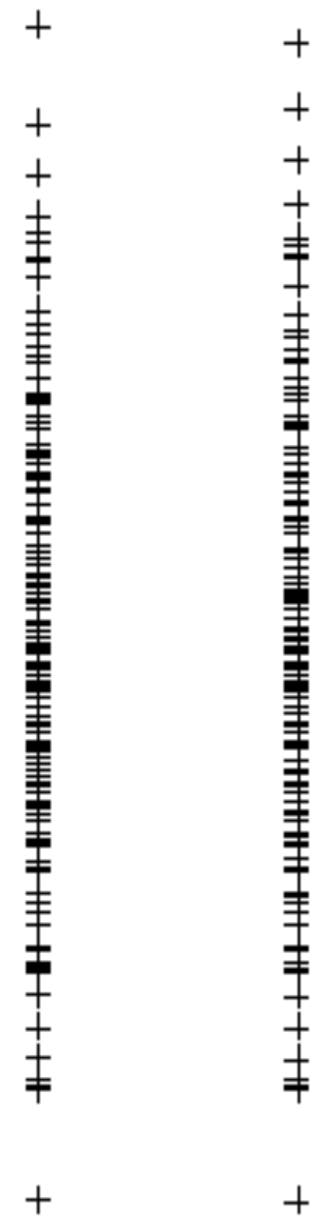
Let's check.

f=K=J/2

L=8

Add interaction term to the Ising Hamiltonian:

$$K\sum_{j=1}^{L-1}\sigma_j^x\sigma_{j+1}^x = -K\sum_{j=1}^{L-1}\psi_{2j-1}\psi_{2j}\psi_{2j+1}\psi_{2j+2}$$



The strong zero mode indeed survives interactions!

Splitting is exponentially small in L.

Not needing disorder gives some hope that strong zero modes in the interacting case can be understood analytically.

The XXZ spin chain is two coupled Majorana chains:

$$H=\sum_{j=1}^{L-1}\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\Delta\sigma_j^z\sigma_{j+1}^z\right)$$

$$=\sum_{j=1}^{L-1}\left(i\psi_{2j-1}\psi_{2j+2}+i\psi_{2j}\psi_{2j+1}-\Delta\psi_{2j-1}\psi_{2j}\psi_{2j+1}\psi_{2j+1}\right)$$

(exchanging x with z in the earlier fermionization –there are multiple ways to fermionize, but all lead to a four-fermi term)

It has a $\mathbb{Z}_2$-ordered gapped phase when $\Delta>1$.

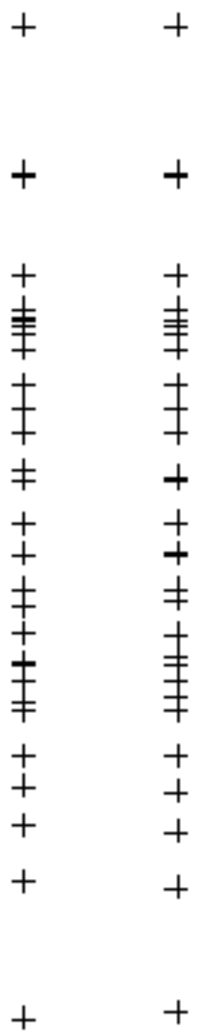
In the limit $\Delta \longrightarrow \infty$, there are the exact strong zero modes σ_1^z, σ_L^z.

These anticommute with spin-flip symmetry $(-1)^F = \prod_{j=1}^{L} \sigma_j^x$

Do they survive at finite Δ?

spin-ordered

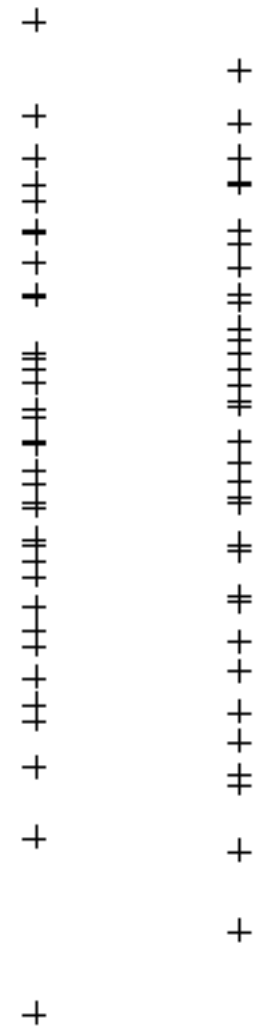
$\Delta = 4$



$(-1)^F =$ 1 -1

Spin-disordered

$\Delta = 1/2$



1 -1

L=8

S_z=0

For other S_z sectors, there are some pairs and some singletons.

I almost have computed the explicit edge zero mode in XXZ by brute force (give me another week).

$$\Psi = \sigma_1^z - \frac{1}{\Delta}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y\right)\sigma_3^z - \frac{1}{\Delta^2}\left(\sigma_1^x\sigma_3^x + \sigma_1^y\sigma_3^y\right)\sigma_4^z$$
$$+\frac{1}{\Delta^2}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y\right)\left(\sigma_3^x\sigma_4^x + \sigma_3^y\sigma_4^y\right)\sigma_5^z + \frac{1}{\Delta^2}\left(\sigma_1 + \sigma_2\right) + \ldots$$

It gets ugly, but much less nasty than you might expect, and much much less nasty than the disordered case.

Presumably the integrability is the reason.

The important question: when is it normalizable?

Using the explicit expression, I find

$$\Psi^2 = 1 + 4\Delta^2 + 10\Delta^4 + 20\Delta^6 + 35\Delta^8 + \ldots$$
$$\to \frac{1}{(1-\Delta^2)^4}$$

This indicates that the strong zero mode transition occurs at $\Delta = 1$ -- the same coupling as the KT transition out of the ordered phase.

Note that Ψ is a very complicated operator, but Ψ^2 is proportional to the identity!

If life were all fermions, one would expect that the strong zero mode goes away when the order in the spin model goes away.

However, we already have good evidence in $\mathbb{Z}_n$-invariant system that the situation is much more interesting!

A parafermionic avatar of the MBL transition

The quantum chain version of the 3-state clock/Potts model:

$$H = -\sum_j \left[f(\tau_j + \tau_j^\dagger) + J(\sigma_j^\dagger \sigma_{j+1} + \mathrm{h.c.}) \right]$$

flip is now "shift"

"clock" potential

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\boxed{\tau\sigma = e^{2\pi i/3}\sigma\tau}$$

Define parafermions just like fermions:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain,

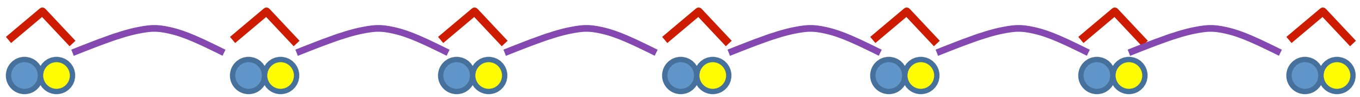
$$\psi_j = \sigma_j \prod_{i<j} \tau_i, \qquad \chi_j = \tau_j \sigma_j \prod_{i<j} \tau_i$$

$$\psi^3 = \chi^3 = 1, \quad \psi^2 = \psi^\dagger, \quad \chi^2 = \chi^\dagger$$

Instead of anticommutators, for $i < j$ and $\gamma = \chi$ or ψ :

$$\gamma_i \gamma_j = e^{2\pi i/3} \gamma_j \gamma_i$$

The Hamiltonian in terms of parafermions:

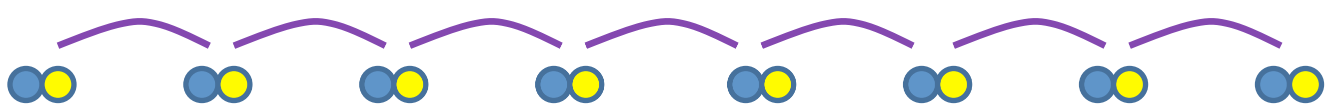


 $= f(\psi_j^\dagger \chi_j + \chi_j^\dagger \psi_j)$  $= J(\psi_{j+1}^\dagger \chi_j + \chi_j^\dagger \psi_{j+1})$

shift term potential

These parafermions are not perturbations of free fermions – they cube to 1. The model isn't even integrable unless $J = f$.

However, when $f = 0$, there are edge zero modes!

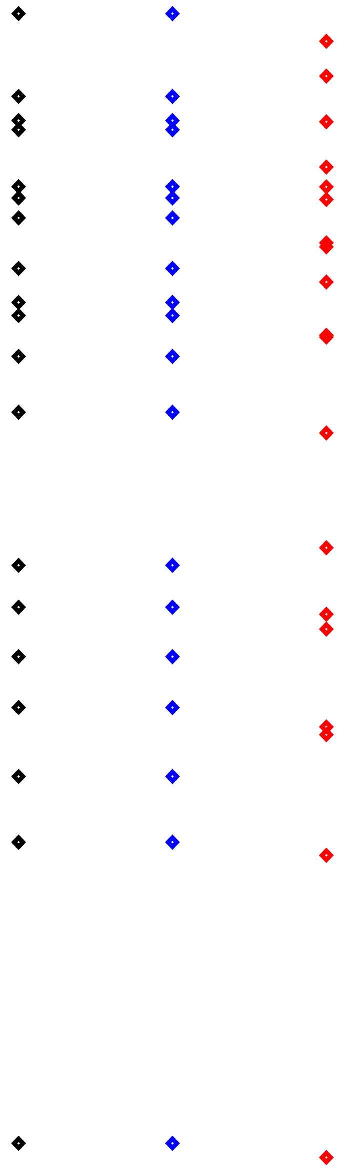


$$[H(f=0), \chi_1] = [H(f=0), \psi_L] = 0$$

Does the strong zero mode remain for $J > f > 0$?

No!

Only weak ones remain

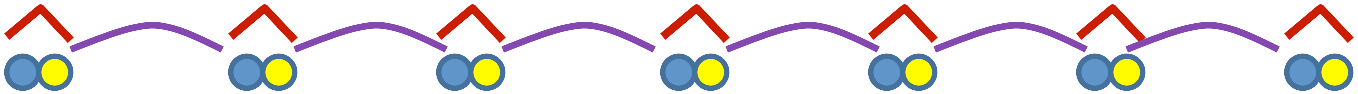


f=J/2

L=4

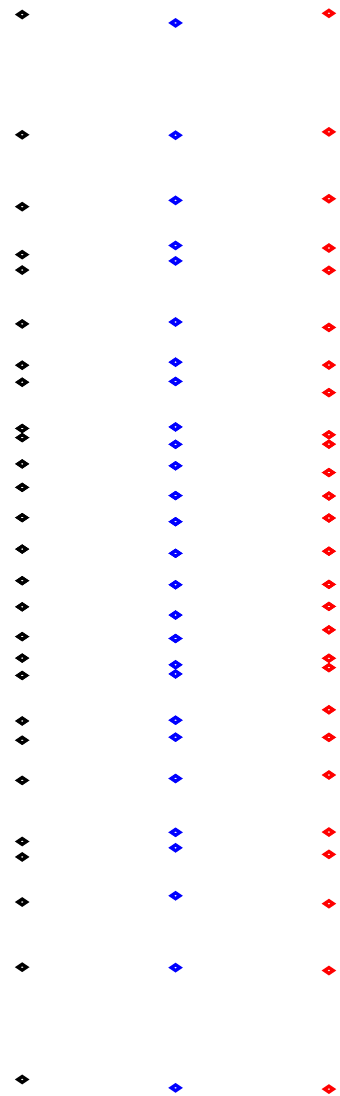
First two are related by S_3 permutation symmetry; not a strong zero mode.

Make the interactions chiral:



 $= f\left(\tau_j e^{i\phi} + \tau_j^\dagger e^{-i\phi}\right)$

 $= J\left(\sigma_j \sigma_{j+1}^\dagger e^{i\theta} + \sigma_j^\dagger \sigma_{j+1} e^{-i\theta}\right)$



A strong zero mode!

f=J/2

L=4

$\phi = \theta = \pi/6$

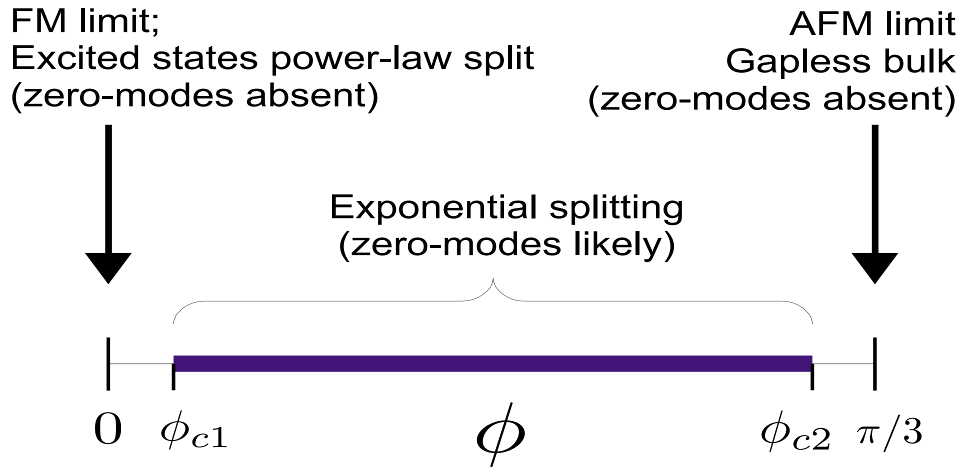
(maximally chiral)

Strong parafermionic zero modes require chirality

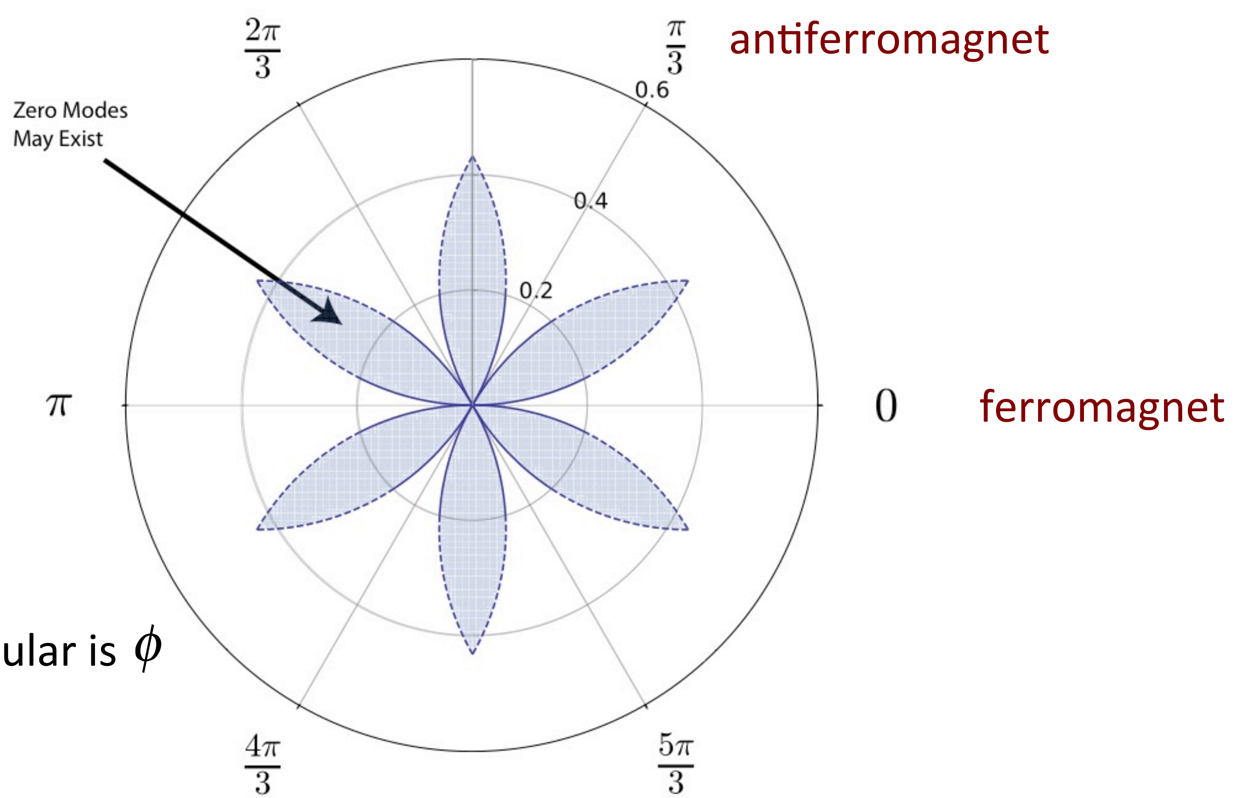
Fendley; Jermyn, Mong, Alicea, Fendley

- Simple perturbative arguments illustrate why zero modes need $\phi \neq 0$.
- The ground state remains degnerate/topologically ordered for any ϕ.
- Thus there is a transition only involving excited states!
- This transition is critical!
- This is an avatar of the MBL transition, an example of an eigenstate phase transition.

At fixed small *f/J*:



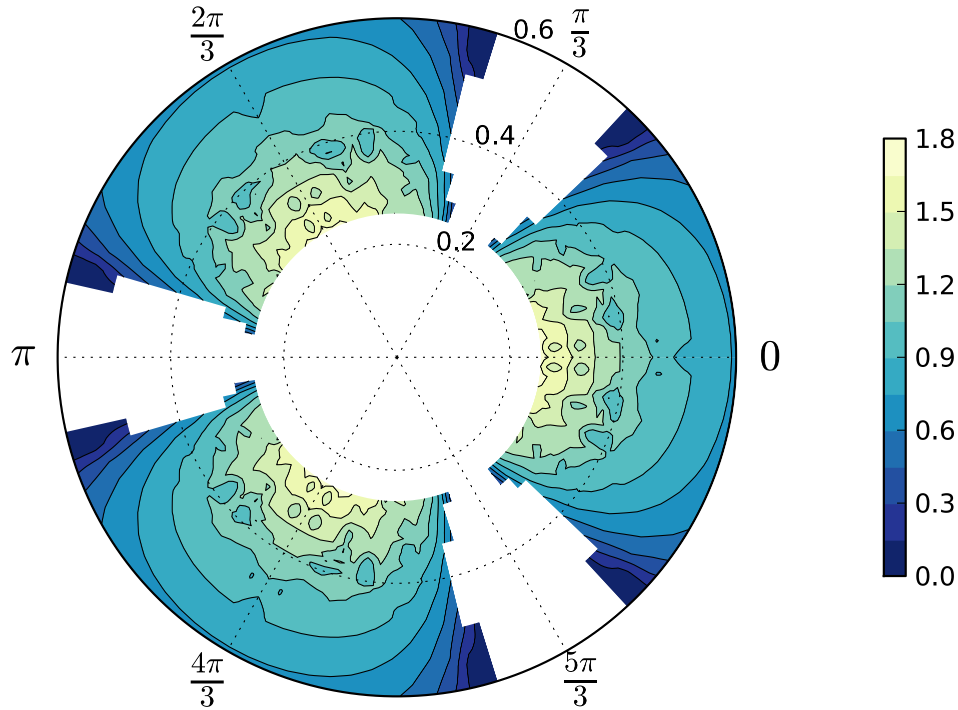
Extrapolating the lowest-order results:



Radial coordinate is *f/J*, angular is ϕ

Splitting from DMRG:

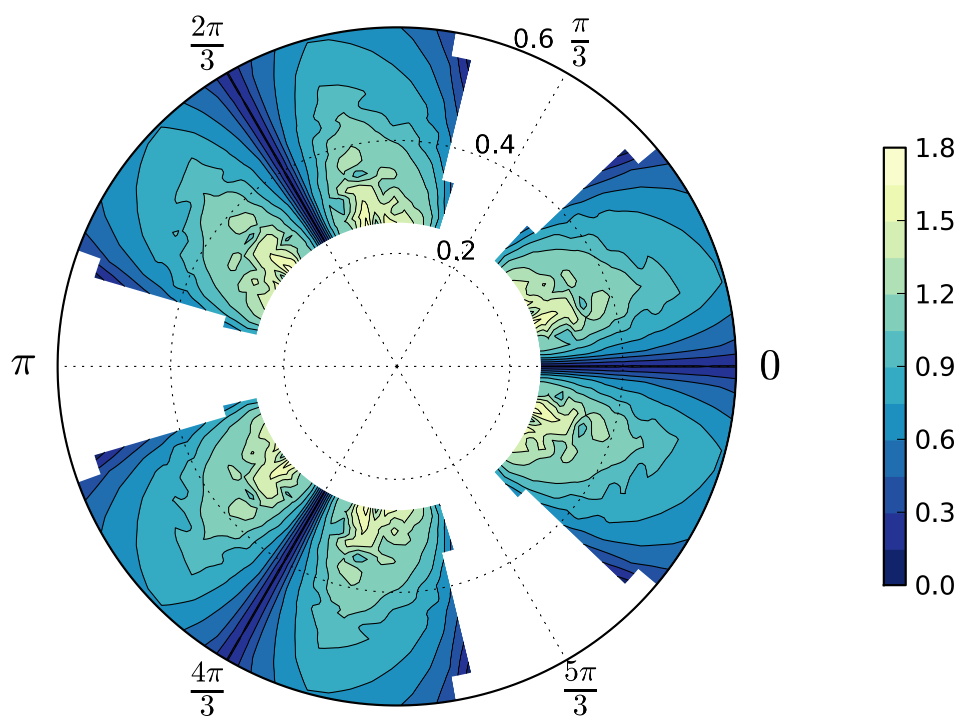
Ground state:



Splitting is $\sim e^{-sL}$

Color represents exponent *s*.

An excited state:



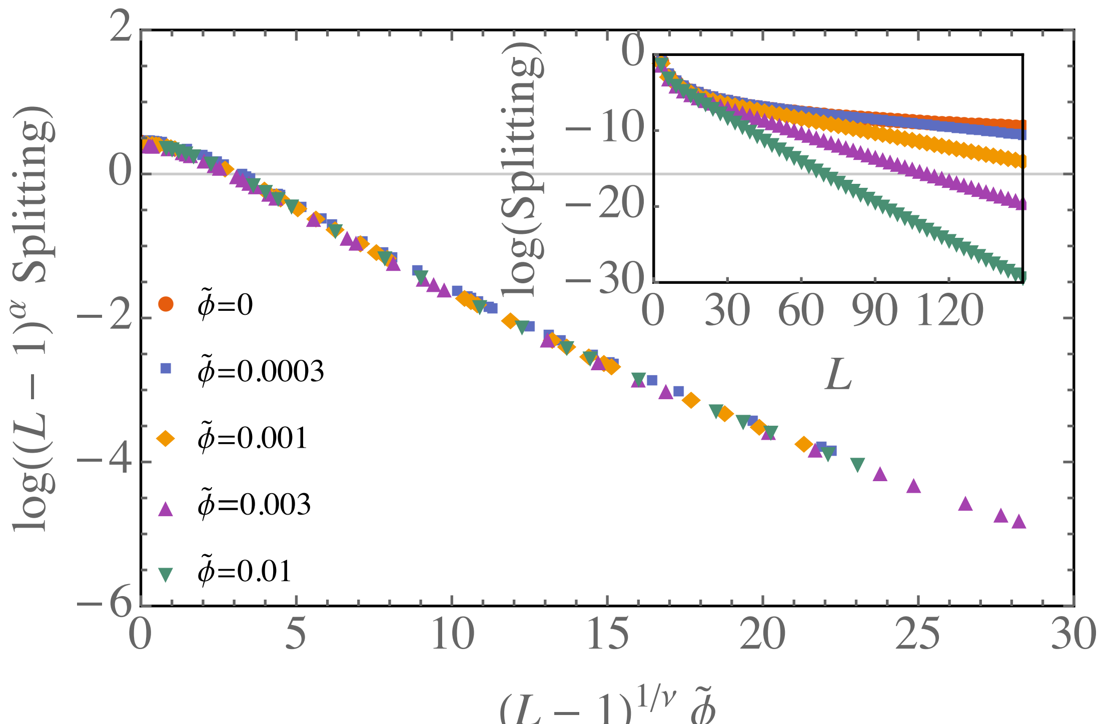
Criticality in the excited-state transition

Near $\phi = \phi_c$, we postulate that in the power-law phase the splitting for a given excited-state multiplet obeys the scaling form

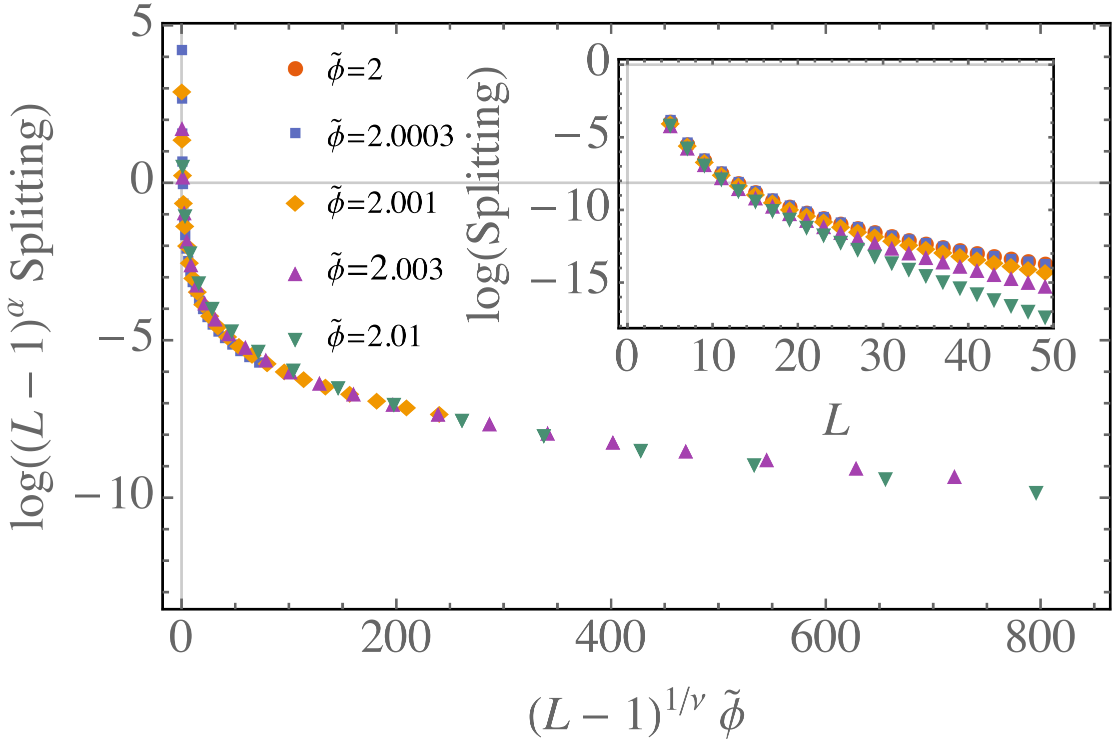
$$\Delta E(\delta\phi, L) = (L-1)^{-\alpha}\,\epsilon((L-1)^{1/\nu}\,\delta\phi)$$

For a low-lying triplet,

$$\alpha = 2,\ \nu = 1/2$$



For a higher-energy triplet:



$$\alpha = 1, \nu \approx .31$$

No conclusion yet

- I need to first finish the computation for XXZ/coupled Majorana chains.

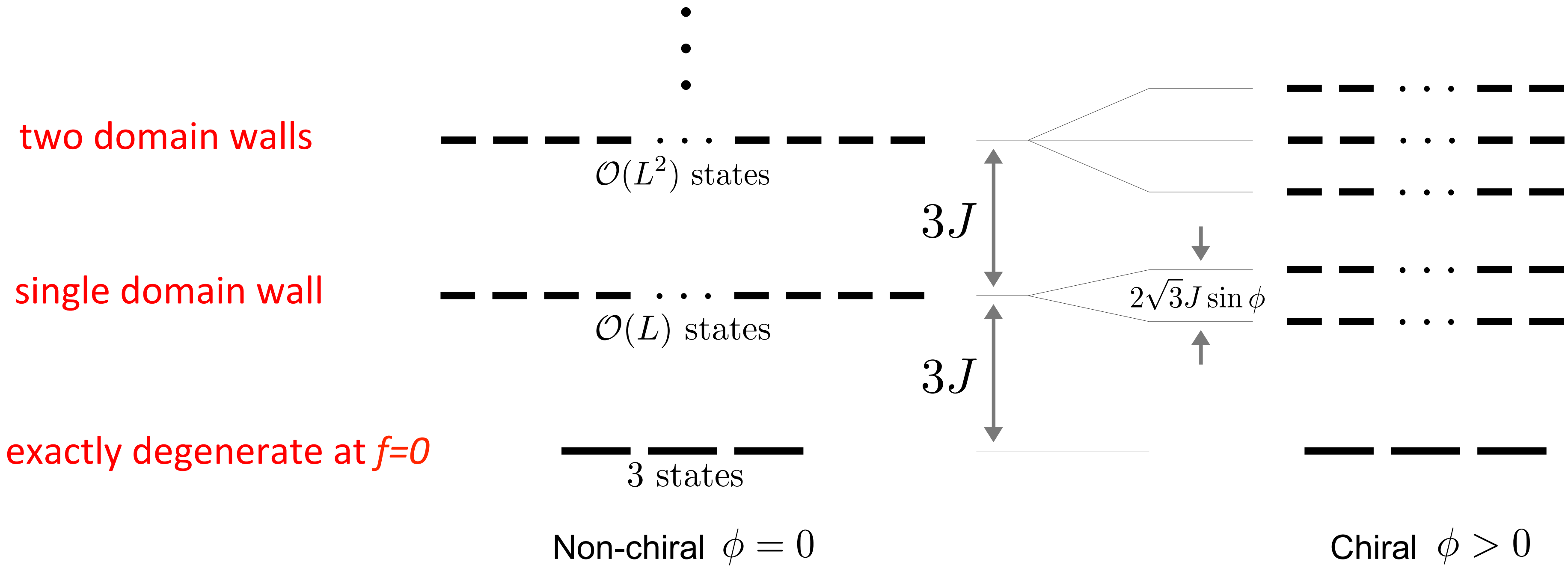
- There is an integrable case of the chiral clock model. This includes the ``superintegrable'' line $\phi = \theta = \pi/6$, halfway between ferromagnet and antiferromagnet.

- In the parafermion case, I gave an all-orders perturbative proof that Ψ exists, but can't prove it's normalizable. The XXZ work suggests that brute force may work for $\phi = \theta = \pi/6$.

  See also Alexandradinata, Regnault, Fang, Gilbert and Bernevig for a thorough analysis of the weak zero modes

- Thus I belive an analytic proof of an eigenstate phase transition in an interacting model is possible.

Spectrum for *f=0* and open boundary conditions:



The ground-state splitting is exponentially small for $f \ll J$ for all ϕ :

$$\Delta E_{\mathrm{g.s.}} \sim f\left(\frac{f}{3J}\right)^{L-1}$$

For a single domain wall at $f \ll J,\ \phi \ll 1$

If $\phi > \phi_{c1} \sim \dfrac{2f}{\sqrt{3}J}$ then the bands arising from $\phi \neq 0$ are far apart.

If $\phi = \phi_{c1}$ then the bands touch and so the states mix.

If $\phi < \phi_{c1}$ then there is power-law splitting. No zero mode!