Mobile Impurity Models and Spin-Charge Separation

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A. Perturbed Luttinger Liquids

large distances

low energies

Lattice model $\longrightarrow$ LL +...

Example: spinless fermions (= spin-1/2 XXZ chain)

$$\boxed{H = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + \text{H.c.}) + V\sum_j n_j n_{j+1}.} \longrightarrow \boxed{\mathcal{H} = \int dx[\mathcal{H}_{\text{LL}}(x) + \mathcal{H}_{\text{irr}}(x)].}$$

$$\mathcal{H}_{\text{LL}}(x) = \frac{v}{16\pi}\left[K(\partial_x\Theta)^2 + \frac{1}{K}(\partial_x\Phi)^2\right],$$

free, compact boson

$$H_{\text{irr}} = \lambda_3^+\partial_x\Phi[(\partial_x\Phi)^2 + (\partial_x\Theta)^2] + \lambda_3^-\partial_x\Phi[(\partial_x\Phi)^2 - (\partial_x\Theta)^2] + \ldots$$

irrelevant perturbations

Projection to low-energy degrees of freedom:

Haldane '82

$$c_j \simeq \sqrt{a_0}\left[Ae^{ik_F x}e^{-\frac{i}{\sqrt{2}}\varphi} + Be^{-ik_F x}e^{\frac{i}{\sqrt{2}}\bar{\varphi}} + \ldots\right],\ x = ja_0$$

lattice spacing

(analogous expressions for spin operators in XXZ)

Ground state correlation functions for XXZ:

$$\langle \mathrm{GS}|S_j^\alpha S_{j+\ell}^\alpha|\mathrm{GS}\rangle \longrightarrow \int \mathcal{D}\phi \left[\sum_\alpha A_\alpha \mathcal{O}_\alpha(0)\right]\left[\sum_\beta A_\beta^* \mathcal{O}_\beta^\dagger(x)\right] e^{-S_{\mathrm{LL}}[\Phi]-S_{\mathrm{irr}}[\Phi]}$$

Perturbation theory in $S_{\mathrm{irr}}[\Phi]$ gives **excellent** agreement with the lattice model result (even for ℓ =1) !

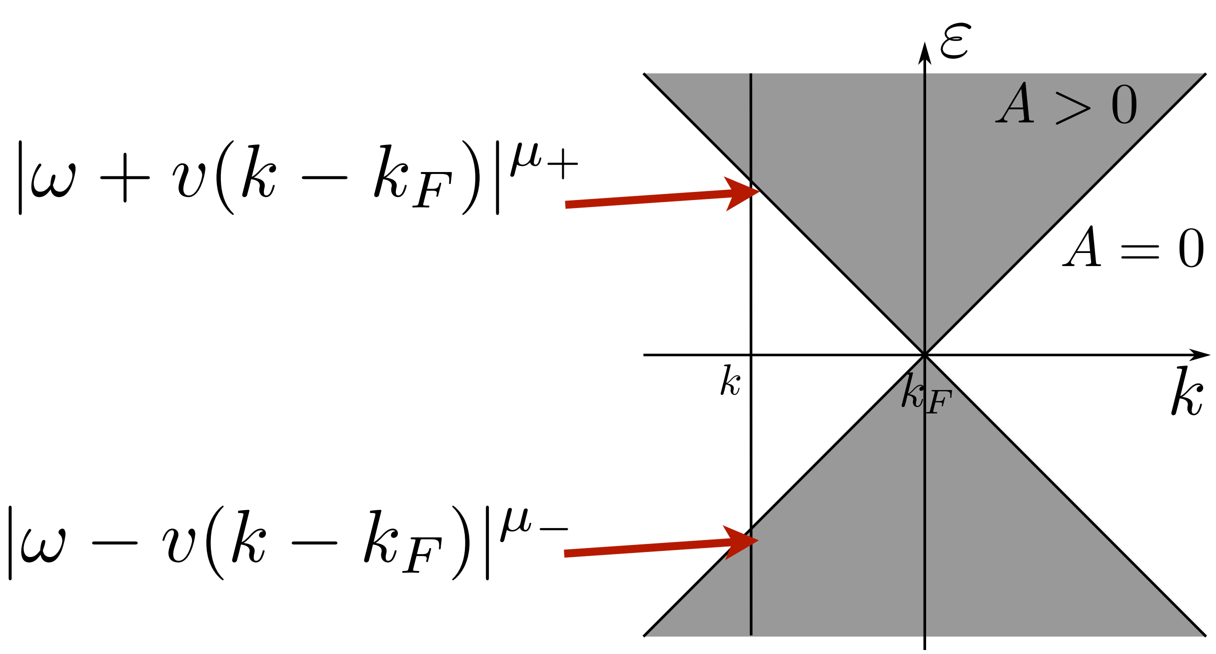
Lukyanov '97

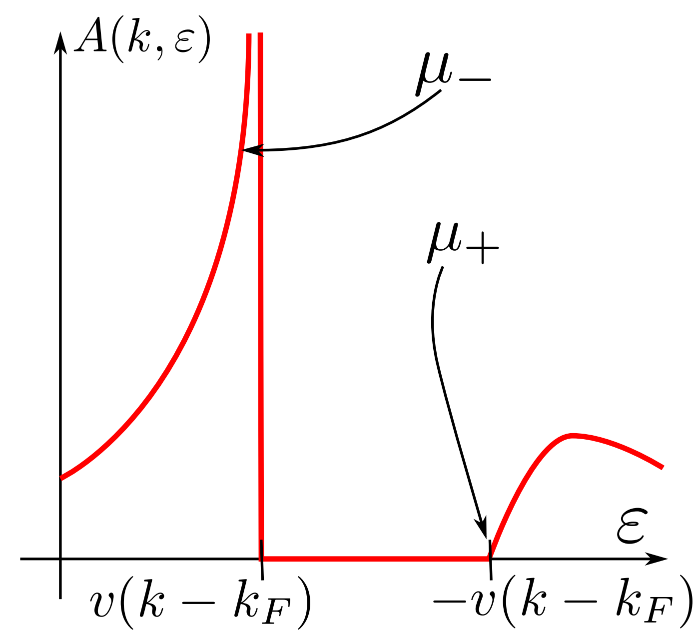
B. Dynamical correlation functions

e.g. single-particle spectral function:

$$A(\omega,k) = -\frac{1}{\pi}\mathrm{Im}\int_0^\infty dt\, e^{i\omega t}\sum_l e^{-ikla_0}\langle \mathrm{GS}| -i\{c_{j+l}(t), c_j^\dagger\}|\mathrm{GS}\rangle$$

Map this to a LL for $\omega \approx 0$ ($\Rightarrow k \approx \pm k_F$):





LL gives power-law threshold singularities

CFT predictions for $\mu_\pm$ are **wrong** !

Why does this happen? **PT in $S_{irr}[\Phi]$ fails.**

Can be understood already for free lattice fermions:

$$H_0 = -t \sum_j c_j^\dagger c_{j+1} + \text{h.c.} \qquad G_{\text{ret}}(\omega, k) = \frac{1}{\omega + i0 - \epsilon(k)}$$

Close to k_F: $\epsilon(k \approx k_F) = v(k - k_F) + \xi(k - k_F)^2 + \ldots$

$$G_{\text{ret}}(\omega, k) = \underbrace{\frac{1}{\omega + i0 - v(k - k_F)}}_{\text{LL result}} \left[1 + \underbrace{\frac{\xi(k - k_F)^2}{\omega + i0 - v(k - k_F)}}_{\text{infrared singularity}} + \ldots \right]$$

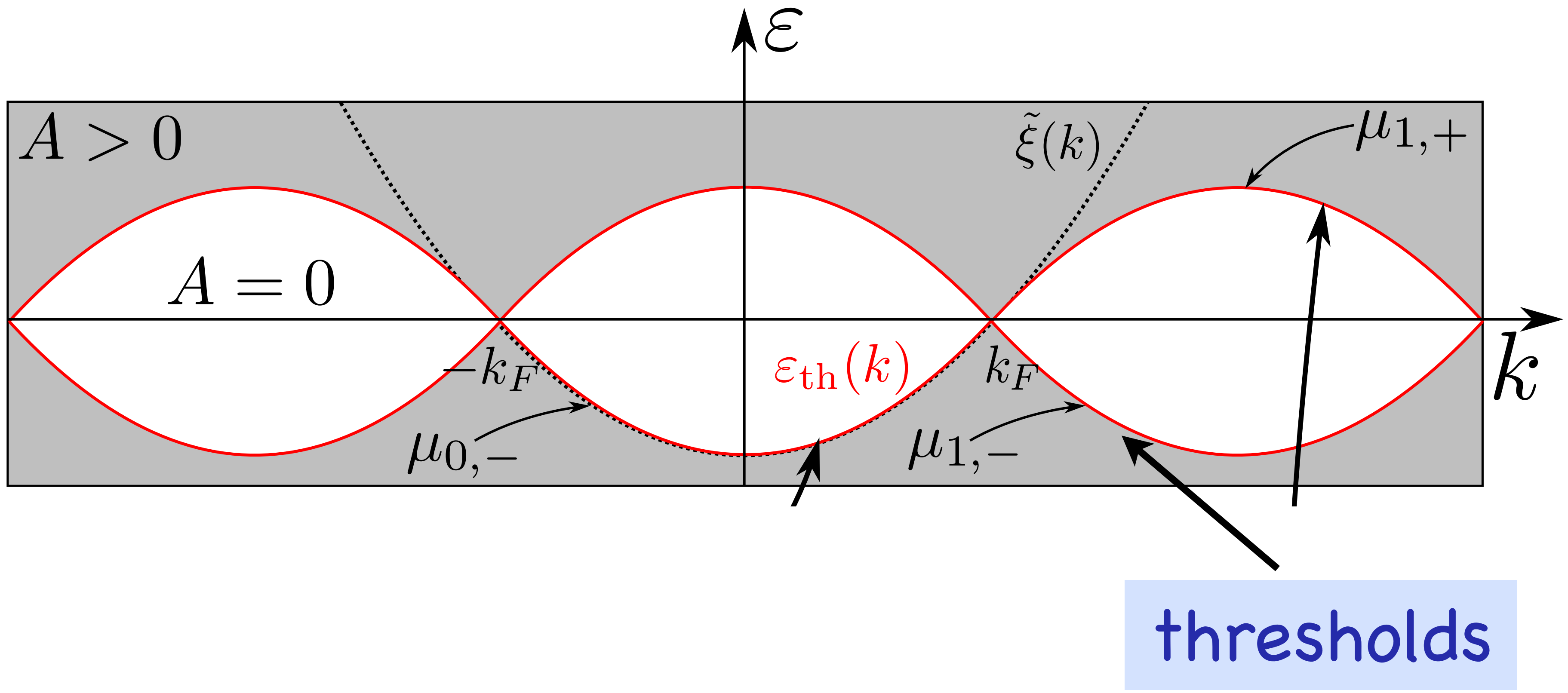
Bad news: must sum PT to all orders, singularities don't simply exponentiate.

There is a neat trick for doing this ....

Pustilnik, Khodas, Kamenev &Glazman '06

C. Mobile impurity models

Bethe Ansatz gives full spectrum of our lattice model:



$$A_<(\omega,k) = 2\pi \sum_n |\langle\psi_n|c_k|\psi_0\rangle|^2 \delta(\omega + E_n - E_0),$$

Kinematics: **just** above the threshold at mtm k only states with a **single** high energy excitation with mtm ≈k contribute

⇒ augment LL by a single "mobile impurity" with mtm ≈k

Impurity has the same quantum numbers as c_j (**small** Δ, Bethe Ansatz)

$$c_j \to \sqrt{a_0}[e^{ik_F x} r(x) + e^{-ik_F x} l(x) + e^{ikx} \chi^\dagger(x)]$$

lattice LL impurity field

Project onto LL & impurity degrees of freedom:

$$\boxed{\begin{aligned}\mathcal{H}_{\rm imp} = \int dx \Big\{ \frac{v}{16\pi} \Big[K(\partial_x \Theta)^2 + \frac{1}{K}(\partial_x \Phi)^2 \Big] + \chi^\dagger(\varepsilon - iu\partial_x)\chi \\ + \chi^\dagger \chi [f(k)\partial_x \varphi + \bar{f}(k)\partial_x \bar{\varphi}] \Big\}\end{aligned}}$$

Close to the threshold (at ω<0) we have

$$\boxed{G_{{\rm ret},<}(\omega,k) \approx -i \int_0^\infty dt\, e^{i\omega t} \int_{-\infty}^{\infty} dx \langle \psi_0 | \chi(0,0) \chi^\dagger(t,x) | \psi_0 \rangle.}$$

The mobile impurity model is solved by a unitary transformation:

$$\varphi^\circ(x) = U\varphi(x)U^\dagger$$
$$\bar{\varphi}^\circ(x) = U\bar{\varphi}(x)U^\dagger$$
$$d(x) = U\chi(x)U^\dagger$$

$$U = e^{-i\int_{-\infty}^{\infty} dx\,[\gamma\varphi(x)+\bar{\gamma}\bar{\varphi}(x)]\chi^\dagger(x)\chi(x)}.$$

$\gamma, \bar{\gamma}$ known functions of $f(k), \bar{f}(k)$

$$\boxed{H_{\mathrm{imp}} = \int dx \Big\{ \frac{v}{16\pi}\Big[K(\partial_x\Theta^\circ)^2 + \frac{1}{K}(\partial_x\Phi^\circ)^2\Big] + d^\dagger(\varepsilon - iu\partial_x)d + \dots \Big\}}$$

impurity decouples in new basis!

Threshold singularity at ω<0:

$$\boxed{A_<(\omega,k) \sim \theta(\epsilon(k)-\omega)|\epsilon(k)-\omega|^{-1+2(\gamma^2+\bar{\gamma}^2)}.}$$

How to fix $\gamma, \bar{\gamma}$???

Pereira, Affleck & White '09

1. Use Bethe Ansatz to calculate finite-size spectrum of lattice Hamiltonian in presence of a single high-energy excitation.

2. Calculate finite-size spectrum of the mobile impurity model using mode expansion.

3. Equate the two results $\Rightarrow$ **exact threshold exponent μ(k).**

confirmed by direct
BA calculation

Kitanine et al '12

D. Mobile impurity models & spin/charge separation

spinful fermions: Hubbard model

$$H_{\mathrm{Hub}} = -t \sum_{j=1}^{L} \sum_{\sigma} (c^\dagger_{j,\sigma} c_{j+1,\sigma} + \mathrm{h.c.}) + U \sum_j n_{j,\uparrow} n_{j,\downarrow},$$

large distances
low energies

$$\mathcal{H} = \sum_{\alpha=c,s} \frac{v_\alpha}{16\pi} \int dx \Big[K_\alpha (\partial_x \Theta_\alpha)^2 + \frac{1}{K_\alpha} (\partial_x \Phi_\alpha)^2 \Big] + \int dx\, \mathcal{H}_{\mathrm{irr}}(x)$$

$$\mathcal{H}_{\mathrm{irr}}(x) = \lambda_1 \cos \Phi_s + \lambda_2 \partial_x \Phi_c \cos \Phi_s + \sum_{\alpha=c,s} \lambda^{\pm}_{3,\alpha} \partial_x \Phi_c \left[\left(\partial_x \Phi_\alpha\right)^2 \pm \left(\partial_x \Theta_\alpha\right)^2 \right] + \ldots$$

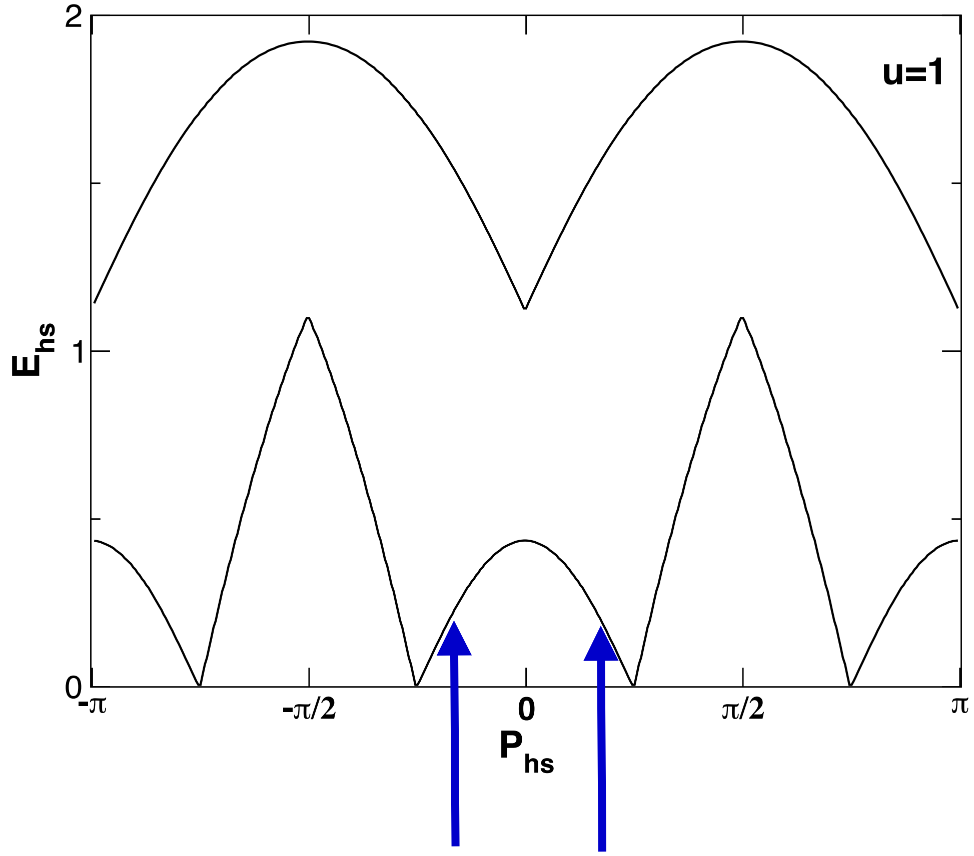
Single-particle spectral function:

$$A(\omega,k) = -\frac{1}{\pi}\mathrm{Im}\int_0^\infty dt\, e^{i\omega t}\sum_l e^{-ikla_0}\langle \mathrm{GS}| - i\{c_{j+l,\sigma}(t), c^\dagger_{j,\sigma}\}|\mathrm{GS}\rangle$$

Probes all excitations with quantum numbers S=1/2, Q=$\pm$e

Bethe Ansatz: excitations at all energies composed of **holons** (Q=$\pm$e, S=0), **spinons** (Q=0, S=$\pm$1/2) (and their bound states).

cf Essler&Korepin '94



threshold

n=0.5

corresponds to a **high-energy spinon + low-energy holon**

Kinematics: **just** above the threshold only states with a **single** high energy excitation contribute

$\Rightarrow$ single mobile spinon impurity!

The main difficulty

Projection onto low-energy and impurity d.o.f. now is

$$c_{j,\uparrow} \simeq \sum_{\alpha} g_{\alpha}(x)\mathcal{O}_{\alpha}^{\mathrm{CFT}}(x) + e^{ikx}\chi^{\dagger}(x)\mathcal{Q}^{\mathrm{CFT}}(x)$$

known

???

Schmidt, Imambekov &Glazman '10

Scheme, in which the impurity is weakly interacting and has fractional quantum numbers at variance with Bethe Ansatz

☞ Essler, Pereira &Schneider '15

E. Field Theory in terms of holons/spinons

bosonized formulation

$$\mathcal{H}=\sum_{\alpha=c,s}\frac{v_\alpha}{16\pi}\int dx\Big[K_\alpha(\partial_x\Theta_\alpha)^2+\frac{1}{K_\alpha}(\partial_x\Phi_\alpha)^2\Big]+\int dx\ \mathcal{H}_{\rm irr}(x)$$

Refermionize in terms of fields carrying only spin/charge:

cf Coleman '75

$$\mathcal{H}=\int dx\left[\mathcal{H}_c(x)+\mathcal{H}_s(x)+\mathcal{H}_{cs}(x)\right],$$

$$\mathcal{H}_c=R_c^\dagger(-iv_c'\partial_x-\eta\partial_x^2)R_c+L_c^\dagger(iv_c'\partial_x-\eta\partial_x^2)L_c+g_{c,0}R_c^\dagger R_cL_c^\dagger L_c+\dots,$$

$$\mathcal{H}_s=R_s^\dagger(-iv_s'\partial_x+i\zeta\partial_x^3)R_s+L_s^\dagger(iv_s'\partial_x-i\zeta\partial_x^3)L_s$$

$$+g_{s,0}R_s^\dagger R_sL_s^\dagger L_s+g_{s,1}(R_s^\dagger L_s+L_s^\dagger R_s)+\dots,$$ massive Thirring

$$\mathcal{H}_{cs}=g_1(R_c^\dagger R_c+L_c^\dagger L_c)(R_s^\dagger L_s+\text{h.c.})+\dots$$

in the Hubbard model these fermions are strongly interacting

Projection of lattice fermion operators:

$$c_\sigma \to \sqrt{a_0}\left[e^{ik_Fx}R_\sigma(x) + e^{-ik_Fx}L_\sigma(x) + \ldots\right]$$

$$R_\uparrow(x) \sim \prod_{\alpha=c,s} \underbrace{R_\alpha(x)\ e^{-\frac{i\pi}{2}\int_{-\infty}^{x} dx'\ Q_\alpha(x')}}_{\mathcal{O}_\alpha(x)} + \ldots$$ fractional JW strings

$$Q_\alpha(x) = R_\alpha^\dagger(x)R_\alpha(x) + L_\alpha^\dagger(x)L_\alpha(x)$$ U(1) charges

Can add interactions in lattice model to make spin/charge fermions **weakly interacting** (break SU(2)!!!)

$$H = H_{\mathrm{Hub}} + \sum_{r\geq 1}\sum_j V_r n_j n_{j+r} + \sum_{r\geq 1}\sum_j \left(J_r \mathbf{S}_j \cdot \mathbf{S}_{j+r} + J_r^z S_j^z S_{j+r}^z\right),$$

F. Mobile impurity model

$$R_s(x) \sim r_s(x) + e^{-iqx}\chi_s^\dagger(x)$$

low energy spinons

"high-energy" spinon impurity

Bosonize low-energy fermions $\quad r_\alpha \sim e^{-\frac{i}{\sqrt{2}}\varphi_\alpha^*(x)},\; l_\alpha \sim e^{\frac{i}{\sqrt{2}}\bar{\varphi}_\alpha^*(x)}$

Mobile impurity model:

$$H = \frac{v_\alpha}{16\pi}\int dx \left[\frac{1}{2K_\alpha}\left(\partial_x \Phi_\alpha^*\right)^2 + 2K_\alpha\left(\partial_x \Theta_\alpha^*\right)^2\right] + \int dx \chi_s^\dagger\left(\varepsilon_s - iu_s\partial_x\right)\chi_s$$

$$+ \int dx\; \chi_s^\dagger \chi_s \left[\sum_\alpha f_\alpha(q)\partial_x\varphi_\alpha^* + \bar{f}_\alpha(q)\partial_x\bar{\varphi}_\alpha^*\right].$$

Projection onto low-energy and impurity d.o.f. becomes

$$c_{j,\uparrow} \simeq \sum_{\alpha} g_\alpha(x)\mathcal{O}_\alpha^{\mathrm{CFT}}(x) + e^{-iqx}\chi^\dagger(x)e^{\frac{i}{4\sqrt{2}}\Phi_s^*(t,x)} + \ldots$$

Conjecture: for strongly interacting spin/charge fermions the same expressions apply, only the parameters need to be adjusted.

Finally

- remove interaction by unitary transformation
- fix parameters of impurity model by comparing FS spectrum to Bethe Ansatz calculation

$\Rightarrow$ exact results for threshold exponents

G. Numerical tests

Translate results for threshold singularities to time domain

$$G(t,k) \sim \sum_{\alpha} A_{\alpha} e^{i\omega_{\alpha} t + \phi_{\alpha}} t^{-\gamma_{\alpha}},$$

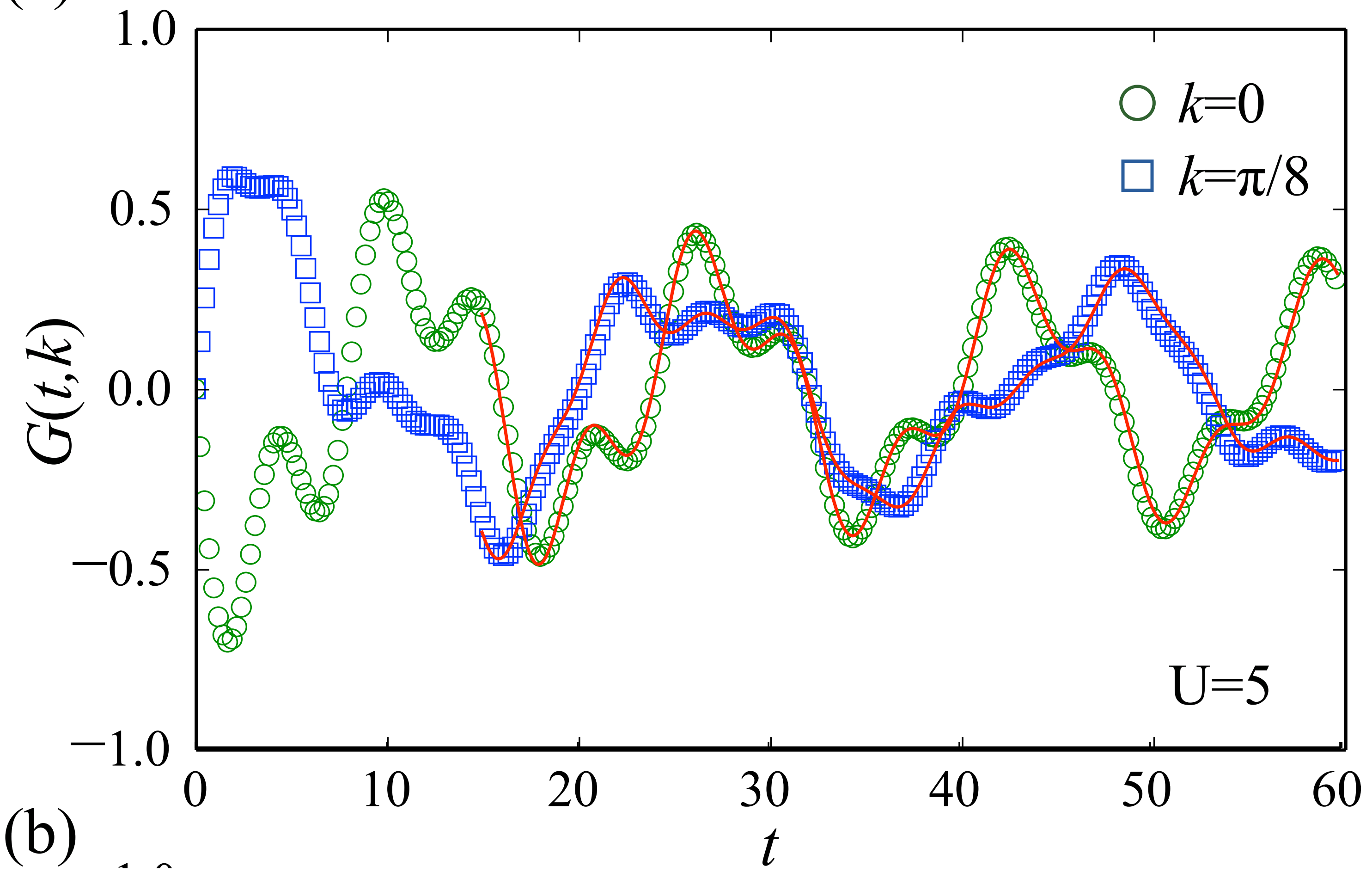
use as fit parameters

exact results

t-DMRG results

Seabra et al '14

(a)



(b)

H. Luther-Emery point for charge and spin

Our construction raises an interesting question:

Is there a lattice model of strongly interacting electrons, that maps exactly to **free fermionic** spinons and holons?

$$H = H_{\rm Hub} + \sum_{r\geq 1}\sum_j V_r n_j n_{j+r} + \sum_{r\geq 1}\sum_j \left(J_r \mathbf{S}_j \cdot \mathbf{S}_{j+r} + J_r^z S_j^z S_{j+r}^z\right) + \ldots$$

$$\Updownarrow$$

$$H = \sum_{\alpha=c,s}\sum_q \epsilon_\alpha(q) d_\alpha^\dagger(q) d_\alpha(q)$$

RG analysis of vicinity of LE point is quite interesting.

Summary

- CFT fails to describe dynamical properties of lattice models even at low energies.
- Nice method to augment CFT by "mobile impurity" d.o.f. to calculate dynamical properties at finite frequencies.
- Spin-charge separated case is difficult, but can be handled.
- MIM mapping works for any two point function of local operators.
- MIMs not always easy to solve.
- Construct lattice model of free fermionic holons and spinons!