



# Non-equilibrium steady states in many-body quantum systems

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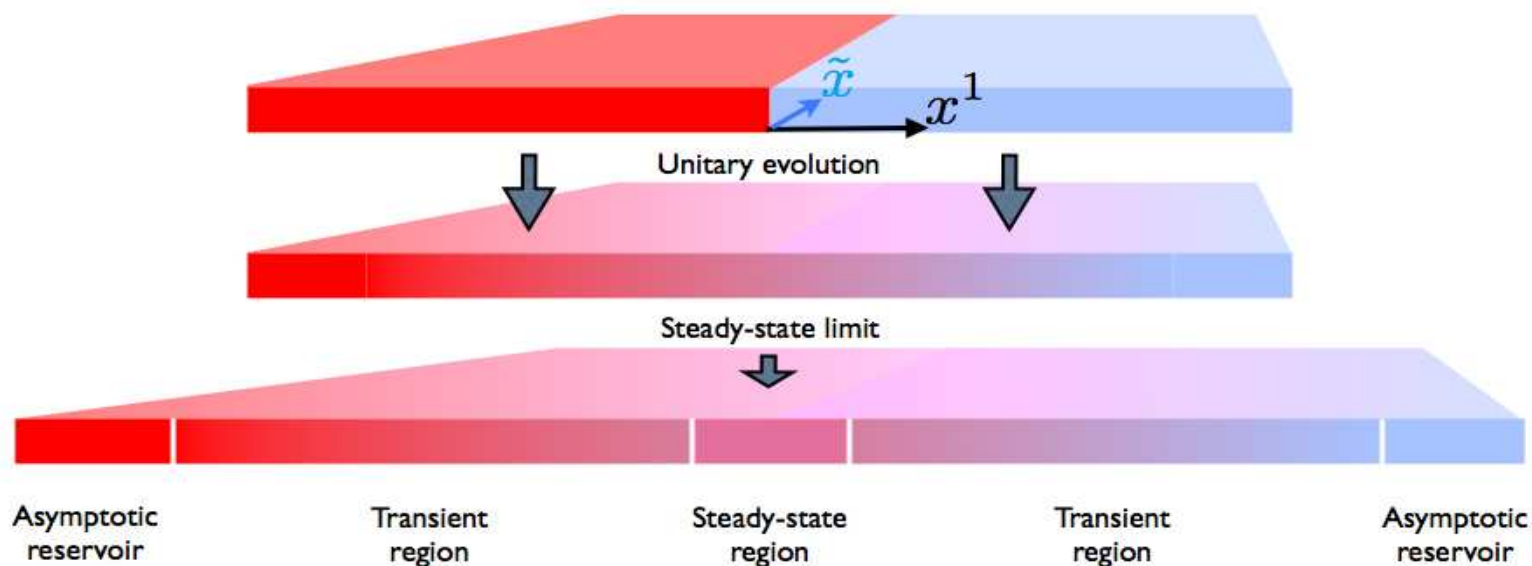
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## Partitioning approach

[Caroli et. al. 1971; Rubin et. al. 1971; Spohn et. al. 1977]

Consider some extended, local many-body quantum system separated into two halves, independently thermalized. Then suddenly connect them (local quench) and wait for a long time (unitary evolution).



**Generically, expect steady state to be trivial: thermalization, no flows.**

In what situation can there be a nontrivial current?

Asymptotic baths very far; steady state translation invariant  $\Rightarrow$  No gradients  $\Rightarrow$  no diffusive transport (cf Fourier's law).

Current emerges in steady-state region iff there is ballistic transport



## Ballistic steady state

- By stationarity and Eigenstate Thermalization Hypothesis [Deutsch 1991, Srednicki 1994, Rigol, Dunjko, Olshanii 2008], steady state described by **(semi-)local conserved charges**.
- By cluster property, steady states is **exponential of local conserved charges** (cf GGE).

Need a parity-odd conserved charge  $P$ :

$$e^{-\beta H + \nu P + \dots}, \quad \langle \mathcal{O} \rangle_{\text{stat}} = \frac{\text{Tr} (e^{-\beta H + \nu P + \dots} \mathcal{O})}{\text{Tr} (e^{-\beta H + \nu P + \dots})}$$

Steady-state limit: only in central region, for local observables,

$$\langle \mathcal{O} \rangle_{\text{stat}} = \lim_{\nu L \gg t \rightarrow \infty} \langle e^{iHt} \mathcal{O} e^{-iHt} \rangle_0, \quad \rho_0 = e^{-\beta_l H_l - \beta_r H_r}, \quad H = H_l + \delta H_{lr} + H_r$$

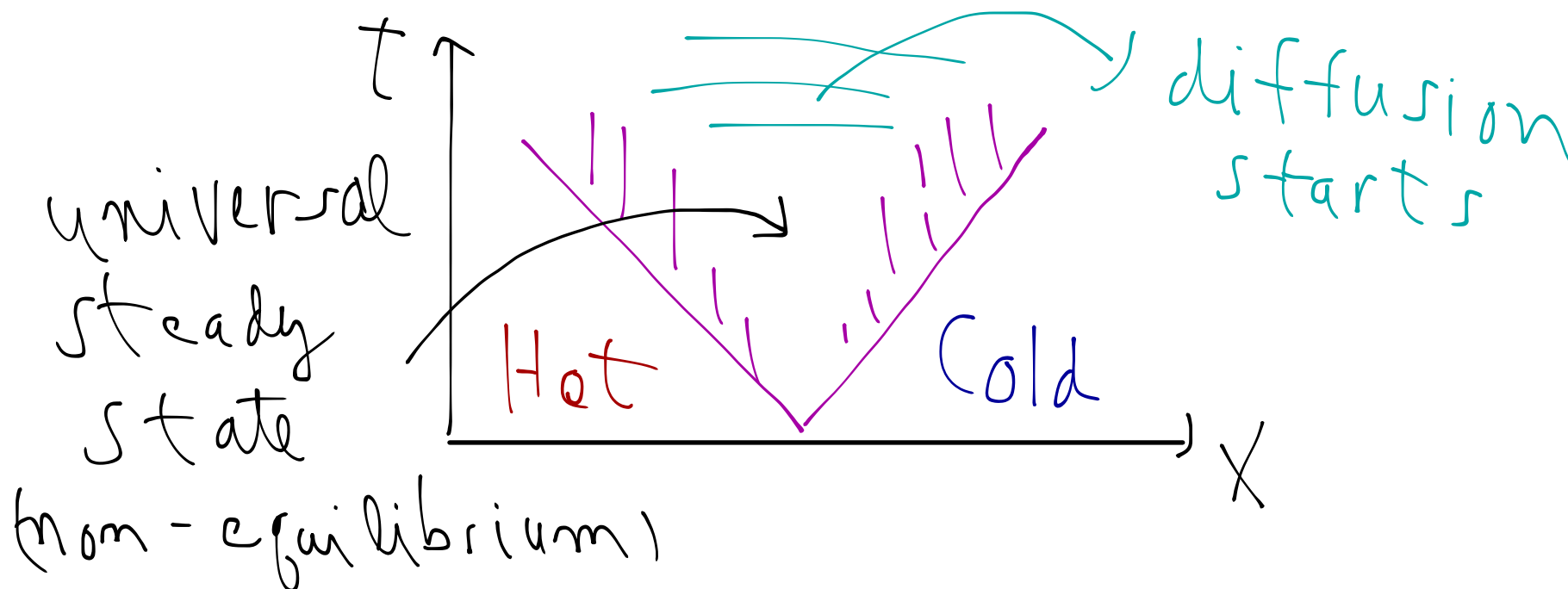
## Near quantum criticality

Near zero-temperature quantum criticality: continuous translation invariance emerges

Momentum  $P$

Universal steady state near criticality, with “diffusion time”  $t_{\text{diff}}(T_1, T_r)$  set by temperatures,

$$\langle O \rangle_{\text{stat}} = \lim_{t_{\text{diff}}(T_1, T_r), \nu L \gg t \rightarrow \infty} \langle e^{iHt} O e^{-iHt} \rangle_0.$$



### If the total current is a conserved quantity

Let  $\underline{j}$  be a **current observable** for transport of quantity  $q$ , i.e.  $\partial_t q + \nabla \cdot \underline{j} = 0$ .

Let  $j := \underline{j}^1$  be longitudinal component, and assume that there is some  $\underline{k}$  such that

$$\partial_t j + \nabla \cdot \underline{k} = 0.$$

$\int d^d x j$  is **conserved**  $\Rightarrow$  nonzero Drude peak, linear-response conductivity

Example: **Lorentz invariant energy transport** ( $z = 1$  near-critical systems),

$$\partial_\mu T^{\mu\nu} = 0 \text{ and } T^{\mu\nu} = T^{\nu\mu}$$

Set  $q = h := T^{00}$ ,  $\underline{j} = \underline{p} := T^{0i}$ ,  $\underline{k} = T^{1i}$ , and we have  $P = \int d^d x j$ .



## Linear response: sound velocity

Take small variations about local Gibbs equilibrium

$$\langle q(x, t) \rangle_0 \approx \langle q \rangle + \delta q(x, t), \quad \langle j(x, t) \rangle_0 \approx \delta j(x, t), \quad \langle k(x, t) \rangle_0 \approx \langle k \rangle + \delta k(x, t)$$

Assume local thermalization: Equation of state  $\langle k \rangle = F(\langle q \rangle)$  valid at every point:

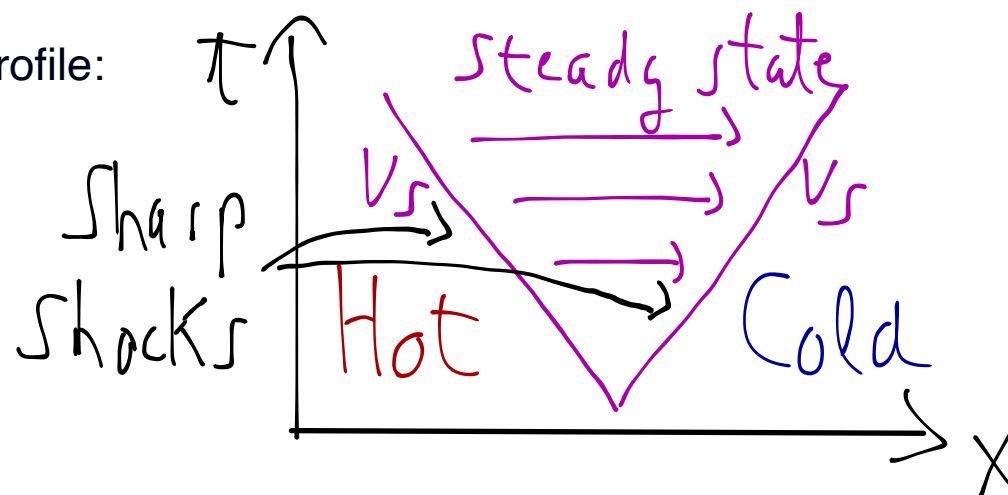
$$\delta k(x, t) = F'(\langle q \rangle) \delta q(x, t)$$

Conservation equations imply wave equation with sound velocity  $v_s = \sqrt{F'(\langle q \rangle)}$ :

$$\delta q(x, t) = f(x - v_s t) + g(x + v_s t), \quad \delta j(x, t) = v_s (f(x - v_s t) - g(x + v_s t))$$

Solving with initial zero-current step profile:

$$\delta j_{\text{stat}} = \frac{\delta k_l - \delta k_r}{2v_s}.$$



## An inequality that quantifies non-equilibrium ballistic transport

[BD 2014]

If “pressure”  $k$  is monotonic on large scales in transient regions, then

$$j_{\text{stat}} \geq \frac{k_l - k_r}{2v}$$

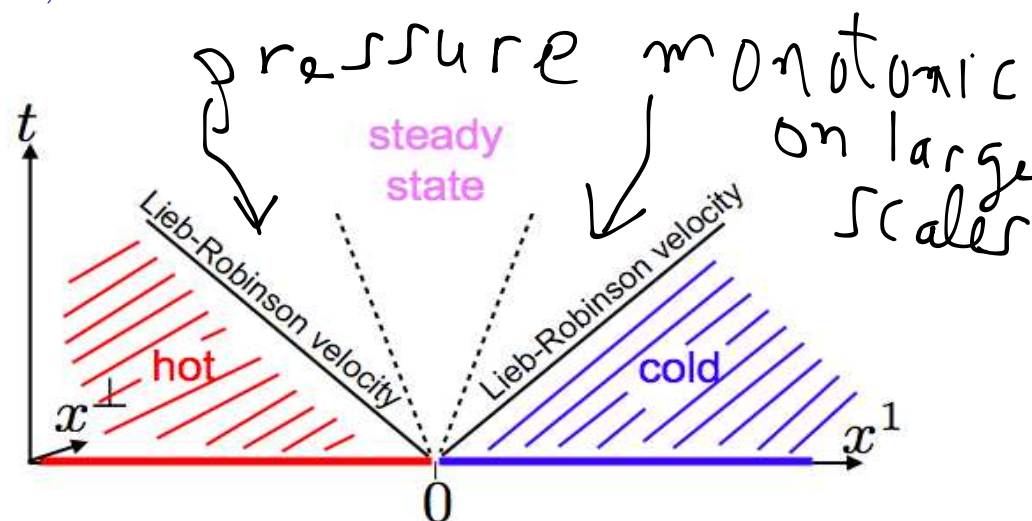
where  $v$  is **Lieb-Robinson velocity** and  $k_{l,r}$  are thermal averages in left and right reservoir.

Can define “transient velocities”:

$$v_{l,r} := \pm \frac{k_{l,r} - k_{\text{stat}}}{j_{\text{stat}}}, \quad v_{l,r} \leq v.$$

From the linear response calculation:

$$\lim_{\text{equilibrium}} v_{l,r} = v_s = \text{sound velocity.}$$



## Shocks

Suppose two-shock picture of linear response remains “mostly true”:  $o(t)$  **transient regions**.

Take integral form of conservation equations through shocks

$$\partial_t \mathbf{h} + \partial_x \mathbf{j} = 0, \quad \partial_t p + \partial_x \mathbf{k} = 0$$

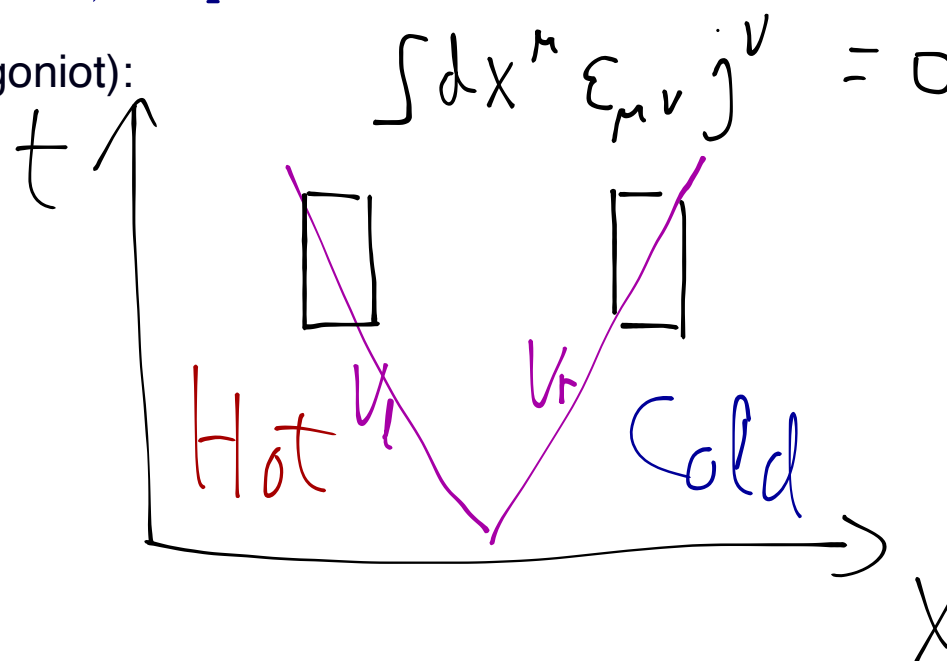
Four connection equations (Rankine-Hugoniot):

$$v_l (\mathbf{h}_l - \mathbf{h}_{\text{stat}}) = \mathbf{j}_{\text{stat}}$$

$$v_l p_{\text{stat}} = \mathbf{k}_l - \mathbf{k}_{\text{stat}}$$

$$v_r (\mathbf{h}_{\text{stat}} - \mathbf{h}_r) = \mathbf{j}_{\text{stat}}$$

$$v_r p_{\text{stat}} = \mathbf{k}_{\text{stat}} - \mathbf{k}_r$$



If we know  $\mathbf{h}(\beta, \nu)$ ,  $\mathbf{j}(\beta, \nu)$ ,  $p(\beta, \nu)$  and  $\mathbf{k}(\beta, \nu)$ :

Four equations, four unknowns  $\beta_{\text{stat}}, \nu_{\text{stat}}, v_l, v_r \Rightarrow$  unique solution (?)

## Relativistic thermodynamics

If  $j = p$  then:

- Stress-energy tensor in state  $e^{-\beta H + \nu P}$   
(with  $\beta = \beta_{\text{rest}} \cosh \theta$ ,  $\nu = \beta_{\text{rest}} \sinh \theta$ ,  $u = \begin{pmatrix} \cosh \theta \\ \sinh \theta \end{pmatrix}$ ):

$$T^{\mu\nu} = k_{\text{rest}} \eta^{\mu\nu} + (\mathbf{h}_{\text{rest}} + k_{\text{rest}}) u^\mu u^\nu$$

where  $k_{\text{rest}} = k(T_{\text{rest}})$ ,  $\mathbf{h}_{\text{rest}} = \mathbf{h}(T_{\text{rest}})$  (thermal averages)

- Temperature dependence in thermal state  $e^{-\beta H}$ :

$$T \frac{d}{dT} k(T) = \mathbf{h}(T) + k(T) \quad (\text{thermal averages})$$

$\Rightarrow$  Thermal equation of state  $k(T) = F(\mathbf{h}(T))$  fixes everything.

$$\log T = \int^{k(T)} \frac{d\ell}{\ell + F^{-1}(\ell)} = \int^{\mathbf{h}(T)} \frac{d\ell F'(\ell)}{\ell + F(\ell)}.$$

Example: conformal relativistic fluid in  $d$  dimensions,  $k(T) = d \mathbf{h}(T)$ .

### Refinement: pure hydrodynamics

- Assume local generalized thermalization:  $\beta = \beta(x, t)$  and  $\nu = \nu(x, t)$ .
- Hydrodynamic equations are
$$\partial_t \mathbf{h}(\beta, \nu) + \partial_x \mathbf{j}(\beta, \nu) = 0, \quad \partial_t \mathbf{p}(\beta, \nu) + \partial_x \mathbf{k}(\beta, \nu) = 0$$
- Solve using step-profile initial condition
- Shocks are weak self-similar solutions

### Further refinement: viscous hydrodynamics, entropy considerations

- Viscosity terms (higher-derivatives)...
- 2<sup>nd</sup> law of thermodynamics (entropy production)...
- Rarefaction waves (other self-similar solutions)...

### Example 1: 1+1-dimensional conformal field theory

[...; Sotiriadis, Cardy 2008; Bernard, BD 2012; ...]

Here  $j = p$  and  $k = h$ . **Right- and left-moving combinations:**

$$h_+ = \frac{h + p}{2} = h_+(x - t), \quad h_- = \frac{h - p}{2} = h_-(x + t).$$

**Same as linear-response calculation!**

$$j_{\text{stat}} = \lim_{t \rightarrow \infty} \langle h_+(-t) - h_-(t) \rangle_0 = \langle h_+ \rangle_l - \langle h_- \rangle_r = \frac{k_l - k_r}{2}.$$

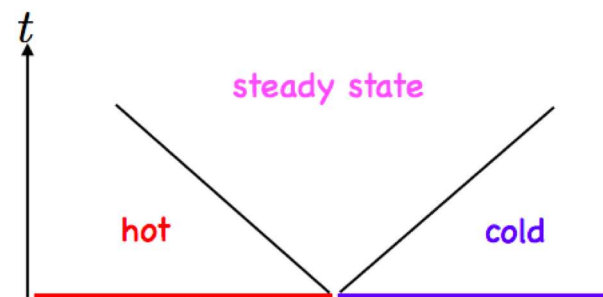
Using CFT results,  $j_{\text{stat}} = \frac{\pi c k_B^2}{12 \hbar} (T_l^2 - T_r^2)$ . Verified numerically [Karrasch, Ilan, Moore 2012] and experimentally [Jezouin, Parmentier, Anthore, Gennser, Cavanna, Jin, Pierre 2013].

**Remarks:**

Inequalities **saturated**,  $v_l = v_r = v_s = v$ .

Sharp shock waves (up to non-universal scales)

Steady state reached “immediately” (idem)

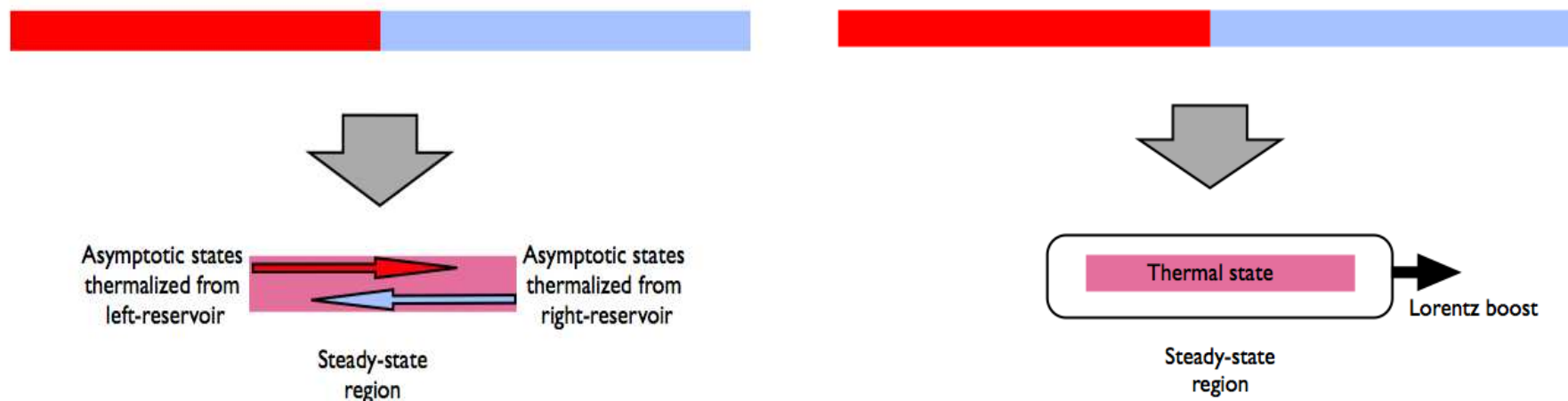


Density matrix for steady state [Bernard, BD 2012; Bhaseen, BD, Lucas, Schalm 2015]:

$$e^{-\beta_1 H_+ - \beta_r H_-} = \exp - \left[ \frac{\beta_1 + \beta_r}{2} H + \frac{\beta_1 - \beta_r}{2} P \right]$$

$H_{\pm}$  = total energy of **right-** / **left-** moving modes;

boost of a thermal state with  $\beta_{\text{rest}} = \sqrt{\beta_1 \beta_r}$ ,  $\tanh \theta = \frac{\beta_r - \beta_1}{\beta_1 + \beta_r}$ .



## Example 2: $T\bar{T}$ -perturbation of CFT

[Bernard, BD in preparation]

$$H = \int dx (T(x) + \bar{T}(x)) + g \int dx T(x)\bar{T}(x).$$

Irrelevant perturbation: low-energy correction to universal behaviour.

Currents at  $O(g)$ :

$$\mathbf{h}(x) = T(x) + \bar{T}(x) + gT(x)\bar{T}(x), \quad \mathbf{p}(x) = T(x) - \bar{T}(x)$$

$$\mathbf{j}(x) = \mathbf{p}(x) + \partial_x(\dots), \quad \mathbf{k}(x) = \mathbf{h}(x) + 2gT(x)\bar{T}(x) + \partial_x(\dots)$$

$\Rightarrow$  Thermodynamics is relativistic:  $\langle \mathbf{j} \rangle = \langle \mathbf{p} \rangle$ , eqn of state  $\langle \mathbf{k} \rangle = \langle \mathbf{h} \rangle + \frac{g}{2} \langle \mathbf{h} \rangle^2$

Can determine exact thermal averages, e.g.

$$\mathbf{h}(T) = \frac{c\pi}{6} T^2 \left( 1 - \frac{gc\pi}{8} T^2 \right).$$



Speed of sound is  $v_s(T) = 1 + \frac{gc\pi}{12} T^2$ , and we find

- Shocks with velocities  $v_l = v_s(T_l)$  and  $v_r = v_s(T_r)$
- Current  $j_{\text{stat}} = \frac{c\pi}{12} (T_l^2/v_l - T_r^2/v_r)$ : still left-right separation in agreement with numerics [Karrasch, Ilan, Moore 2012]
- Steady state density matrix with  $T_{\text{rest}} = \sqrt{T_l T_r} \left(1 - \frac{gc\pi}{48} (T_l - T_r)^2\right)$  and  $\tanh \theta = \frac{T_l - T_r}{T_l + T_r} \left(1 - \frac{gc\pi}{12} T_l T_r\right)$ ; we still have  $\beta = \frac{\beta_l + \beta_r}{2}$
- Shocks of sublinear extent  $O(t^{1/3})$  (conjecture)
- Generic approach  $O(1/\sqrt{t})$  (conjecture)

### Example 3: free higher-dimensional QFT (massive Klein-Gordon model)

Steady state can be described by **independently thermalizing right- and left-movers** (modes with positive and negative longitudinal momenta) **with left and right temperatures**

[Spohn, Lebowitz 1977; ...; Collura, Martelloni 2014; BD, Lucas, Schalm, Bhaseen 2014]

$$e^{-\beta_l H_+ - \beta_r H_-}, \quad H_{\pm} = \int_{p^1 \gtrless 0} d^d p \sqrt{p^2 + m^2} A^{\dagger}(p) A(p)$$

Equivalently [BD, Lucas, Schalm, Bhaseen 2014]:

$$\exp - \left[ \frac{\beta_l + \beta_r}{2} H + \frac{\beta_l - \beta_r}{2} (P_1 + Q) \right]$$

with **semi (or non?)-local parity-odd conserved charge**

$$Q = \int d^d x d^d y : \phi(x) \pi(y) : Q(x - y), \quad Q(x - y) \stackrel{\text{at } d=1}{\sim} -\frac{m}{\pi(x^1 - y^1)}$$

[BD, Lucas, Schalm, Bhaseen 2014]

Current and pressure (here at  $m = 0$ ):

$$\mathbf{j}_{\text{stat}} = d \Gamma(d/2) \zeta(d+1) / (2\pi^{\frac{d}{2}+1}) (T_l^{d+1} - T_r^{d+1}),$$

$$\mathbf{k}_{l,r} = \Gamma((d+1)/2) \zeta(d+1) / (\pi^{\frac{d+1}{2}}) T_{l,r}^{d+1}$$

Remarks:

Inequality ok:  $2\mathbf{j}_{\text{stat}} > \mathbf{k}_l - \mathbf{k}_r$ .

Equilibrium limit **does not give the sound velocity**,  $\lim_{\text{equilibrium}} v_{l,r} \neq 1/\sqrt{d}$ . Signal of generalized Gibbs thermalization (GGE fluid...).

**No shock waves**, rather large transition regions.

Generic approach to steady state is either  $O(1/\sqrt{t})$  or  $O(1/t)$  depending on initial boundary conditions at  $x^1 = 0$ .

### Example 4: non-integrable higher-dimensional CFT

[Bhaseen, BD, Lucas, Schalm 2015; in preparation]

Relativistic system, thermal eqn of state  $k(T) = d h(T)$ . Pure conformal hydrodynamics:

$$\langle T^{\mu\nu}(x, t) \rangle \approx a_d T_{\text{rest}}^{d+1}(x, t) (\eta^{\mu\nu} + (d+1)u^\mu(x, t)u^\nu(x, t)), \quad \partial_\mu \langle T^{\mu\nu} \rangle = 0$$

( $a_d$ : model-dependent normalization) with initial conditions:

$$T_{\text{rest}}(x, 0) = \left\{ \begin{array}{ll} T_l & (x^1 < 0) \\ T_r & (x^1 > 0) \end{array} \right\}, \quad \theta(x, 0) = 0.$$

Assuming two shocks [Bhaseen, BD, Lucas, Schalm 2015; Chang, A. Karch and A. Yarom 2014]:

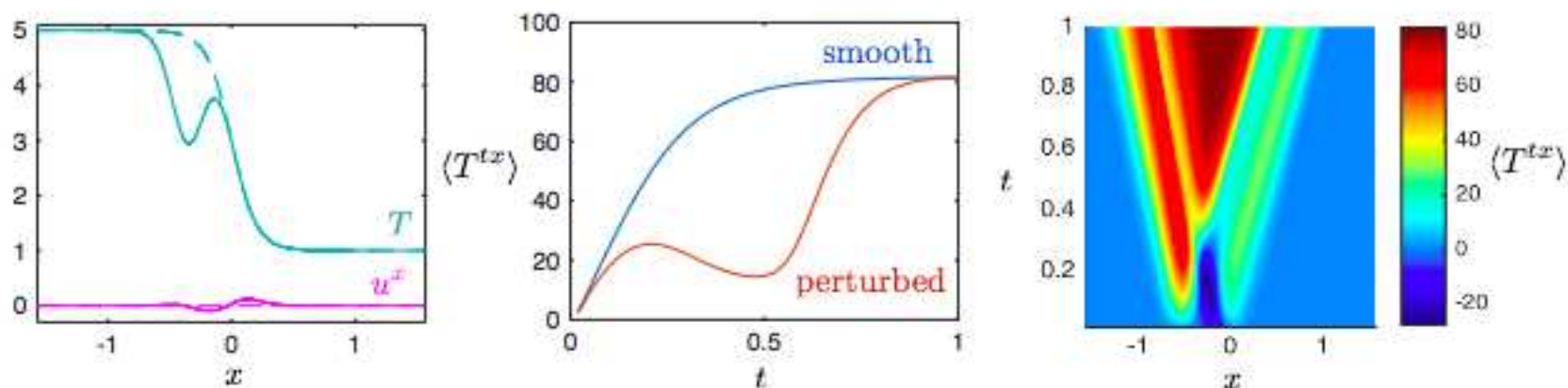
$$v_l = \frac{1}{d} \sqrt{\frac{\tau_l + d\tau_r}{\tau_l + d^{-1}\tau_r}}, \quad v_r = \sqrt{\frac{\tau_l + d^{-1}\tau_r}{\tau_l + d\tau_r}}, \quad \tau_{l,r} = T_{l,r}^{\frac{d+1}{2}}$$

$$T_{\text{rest}} = \sqrt{T_l T_r}, \quad \tanh \theta = \frac{\tau_l - \tau_r}{\sqrt{(\tau_l + d\tau_r)(\tau_l + d^{-1}\tau_r)}}$$

$$\mathbf{j}_{\text{stat}} = \frac{da_d}{d+1} (\tau_l - \tau_r) \sqrt{(\tau_l + d\tau_r)(\tau_l + d^{-1}\tau_r)}.$$

Remarks:

- Emerges naturally under a wide range of initial smoothed-out conditions.

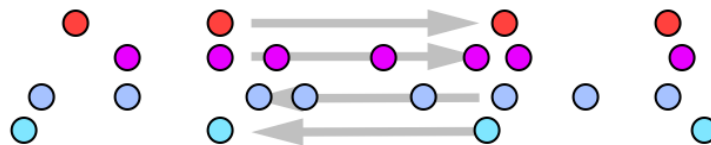


- Verified by AdS/CFT numerics [I. Amado, A. Yarom 2015]
- Inequalities are satisfied:  $v_{l,r} < 1$ .
- Equilibrium limit gives the sound velocity,  $\lim_{\text{equilibrium}} v_{l,r} = 1/\sqrt{d}$ .

## Conclusions

- Hydrodynamics gives general framework in non-integrable models, including perturbed CFT. Integrable models: generalized Gibbs thermalization, generalized hydro?
- Fluctuations and fluctuation relations: **Poisson process interpretation** for energy transport. For instance in 1+1-dimensional CFT: [Bernard, BD 2012; Bernard, BD 2014, BD, Hoogeveen, Bernard 2014]:  $F(z) := \sum_{n=1}^{\infty} c_n \frac{z^n}{n!}$

$$F(z) = \int dq \omega(q) (e^{zq} - 1), \quad \omega(q) = \frac{c\pi}{12} e^{-\beta_{1,r}|q|} \quad (q > 0, q < 0).$$



- Charge transfer (in one dimension [Gutman, Gefen, Mirlin 2010; Bernard, BD 2014]); presence of impurities (in 1-d CFT [Bernard, BD, Viti 2014]); other dynamical exponents; curved connection hypersurface; integrable massive QFT (conjecture [Castro-Alvaredo, Chen, BD, Hoogeveen 2014]); integrable spin chains (conjecture [De Luca, Viti, Mazza, Rossini 2014]); entanglement evolution (in 1-d CFT [Hoogeveen, BD 2015]);...