LCDQM-15 Low Dimensional Quantum Condensed Matter workshop,
University of Amsterdam, July 2, 2015

Geometrodynamics of the Fractional Quantum Hall Effect

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- An effective field theory for the incompressible FQH fluid that describes its gap and long-wavelength collective excitations
- The essential local fields are a flow-velocity, a polarization, and an emergent metric.

- in case I run out of time, I will quickly show the effective action that is finally obtained.

- three dynamical ingredients $g_{ab},\ v^a, P^a$:
  - a "dynamic emergent 2D spatial metric" $g_{ab}(\boldsymbol{x},t)$ with $g \equiv \det\ g$, and Gaussian curvature current $J^\mu_g = \epsilon^{\mu\nu\lambda}\partial_\nu\omega_\lambda(\boldsymbol{x},t)$
  - a flow velocity field $v^a(\boldsymbol{x},t)$
  - an electric polarization field $P^a(\boldsymbol{x},t)$
  - a composite boson current $J^\mu_b = \sqrt{g(\boldsymbol{x},t)}\,(\delta^\mu_0 + v^a(\boldsymbol{x},t)\delta^\mu_a)$

here a is a 2D spatial index, and μ is a (2+1D) space-time index. The fluid motion is non-relativistic relative to the preferred inertial rest frame of the crystal background

- effective bulk action: $\sigma_H = \dfrac{(pe)^2}{2\pi\hbar K}$

$$S = \int d^2x dt\, \mathcal{L}_0 - \mathcal{H}$$

$$\mathcal{L}_0 = \frac{\hbar}{4\pi}\epsilon^{\mu\nu\lambda}\left(K b_\mu \partial_\nu b_\lambda + \beta\omega_\mu\partial_\nu\omega_\lambda\right)$$
$$+ J_b^\mu\left(\hbar(\partial_\mu\varphi - b_\mu - S\omega_\mu) + peA_\mu\right)$$

$U(1)$ Chern-Simons field

$U(1)$ condensate field

"spin connection" of the metric

$$\mathcal{H} = \sqrt{g}\left(\varepsilon(\boldsymbol{v}, B) - U(g, B, P) - (E_a + \epsilon_{ab}v^b B)P^a\right)$$

kinetic energy of flow

metric-dependent correlation energy

In 1983, Laughlin's wavefunction was a unexpected gift to physics that seemed to emerge fully-formed from the void .....

(actually, this seems to be its genesis..)

PRB **256 citations**

Quantized motion of three two-dimensional electrons in a strong magnetic field (/prb/abstract/10.1103/PhysRevB.27.3383)

R. B. Laughlin

Phys. Rev. B **27**, 3383 (1983) - Published 15 March 1983

We have found a simple, exact solution of the Schrödinger equation for three two-dimensional electrons in a strong magnetic field, given the assumption that they lie in a single Landau level. We find that the interelectronic spacing has characteristic values, not dependent on the form of the interaction, which change discontinuously as pressure is applied, and that the system has characteristic excitation energies of approximately $0.03\frac{e^2}{a_0}$, where a_0 is the magnetic length.



- It was quickly confirmed to be the solution to the FQHE mystery
- but WHY it works has never really been explained

- Essentially all the key ideas in the FQHE have emerged as interpretions and generalizations of Laughlin's wavefunction:

flux attachment topological order
composite particles braiding statistics shift
fractional charge chiral cft edge states
conformal block wavefunctions

- I will describe a new insight from a feature of Laughlin's wavefunction that remained undetected for 25 years:

emergent dynamical geometry

FDMH
arXiv:0906-1854
PRL:107 116801 (2011)
arXiv:1112-0990

2D Galilean-invariant Landau levels, uniform B

$$\frac{|\boldsymbol{p} - e\boldsymbol{A}|^2}{2m} = \tfrac{1}{2}\hbar\omega_c \left(a^\dagger a + a a^\dagger\right)$$

$$a^\dagger = \tfrac{1}{2} z^* - \frac{\partial}{\partial z}$$
$$a = \tfrac{1}{2} z + \frac{\partial}{\partial z^*}$$
$$[a, a^\dagger] = 1$$

- mapping to the complex plane

$$z = \frac{(x + iy)}{\sqrt{2}\ell_B}$$

- Lowest Landau level wavefunctions

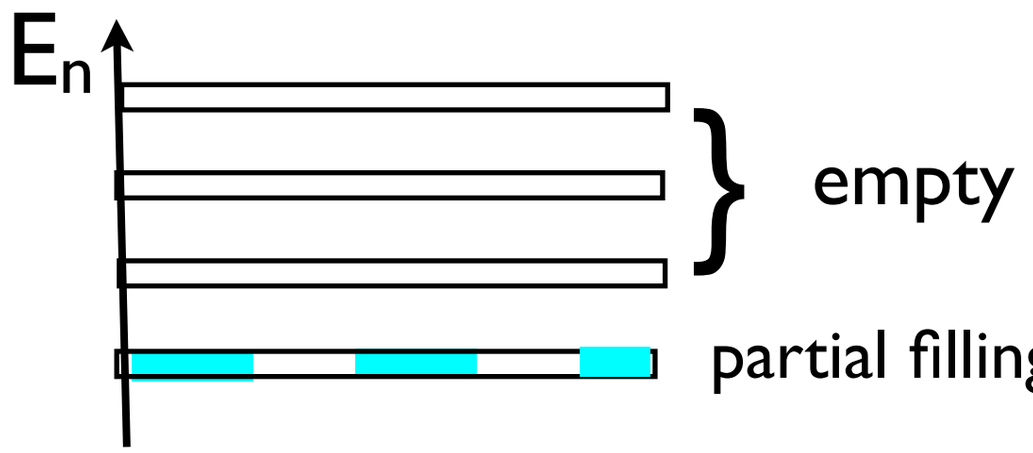
holomorphic

$$a\Psi = 0 \longrightarrow \Psi = f(z) e^{-\frac{1}{2} z^* z}$$

- filled Lowest Landau level (vandermonde determinant)

$$\Psi = \prod_{i<j}(z_i - z_j) \prod_i e^{-\frac{1}{2} z_i^* z_i}$$

uncorrelated



- Laughlin state ($m > 1$)

$$\Psi_L^m = \prod_{i<j}(z_i - z_j)^m \prod_i e^{-\frac{1}{2} z_i^* z_i}$$ highly correlated

bosons for even m, fermions for odd m

$$\nu = \frac{1}{m}$$

some incorrect "conventional wisdom" about the Laughlin state

$$\Psi_L^m = \prod_{i<j} (z_i - z_j)^m \prod_i e^{-\frac{1}{2} z_i^* z_i}$$

- "it is holomorphic (times a Gaussian) because it is a lowest Landau level state" ✗ wrong
- "It is 2D rotationally invariant because the Landau orbits are circular". (angular momentum: $L = \frac{1}{2} mN(N-1)$) ✗ wrong
- "It has no continuously-variable variational parameters". ✗ wrong

The probability that Laughlin himself may have believed these things does not make them true!

- Guiding centers of Landau levels obey the algebra

$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab} \qquad [R^a, (p - eA)_b] = 0$$

Antisymmetric
2D Levi-Civita symbol

$$b^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B} = \frac{1}{2}z - \frac{\partial}{\partial z^*} \qquad a^\dagger = \frac{1}{2}z^* - \frac{\partial}{\partial z}$$

$$b = \frac{R^x - iR^y}{\sqrt{2}\ell_B} = \frac{1}{2}z^* + \frac{\partial}{\partial z} \qquad a = \frac{1}{2}z + \frac{\partial}{\partial z^*}$$

$$b^\dagger f(z) e^{-\frac{1}{2}z^* z} = z f(z) e^{-\frac{1}{2}z^* z}$$ ← action on LLL states

- A new look at the Laughlin state:

$$|\Psi_L^m(\tilde{g})\rangle = \prod_{i<j}(b_i^\dagger - b_j^\dagger)^m |\Psi_0(\tilde{g})\rangle \qquad b_i|\Psi_0(\tilde{g})\rangle = 0$$

$$\varepsilon(\boldsymbol{p}_i - a\boldsymbol{A}(x_i))|\Psi_0(\tilde{g})\rangle = E_n|\Psi_0(\tilde{g})\rangle$$

unimodular metric $\det \tilde{g} = 1$

$$L(\tilde{g}) = \tilde{g}_{ab}\Lambda^{ab} = \frac{1}{2}\sum_i (b_i^\dagger b_i + b_i b_i^\dagger)$$

$$L((\tilde{g})|\Psi_L^m\rangle = (\tfrac{1}{2}mN^2 - \tfrac{1}{2}(m-1)N)|\Psi_L^m\rangle$$

superextensive "statistical spin" — extensive "geometrical spin"

$$\Lambda^{ab} = \frac{1}{4\ell_B^2}\sum_i (R_i^a R_i^b + R_i^b R_i^a)$$

generator of linear area-preserving distortions

- Changes from the original Laughlin formulation:

$$\cancel{\frac{\delta^{ab} p_a p_b}{2m}} \rightarrow \varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

no need for Galilean invariance
(not a property of electrons in solids)

any Landau level will do, not just the "lowest"

$$\cancel{\delta_{ab}} \rightarrow \tilde{g}_{ab}$$

the Laughlin state is PARAMETRIZED by a unimodular emergent 2D spatial metric that SHOULD NOT be identified with the Euclidean metric of Galilean invariance

$\tilde{g}_{ab}$ is also NOT related to an intrinsic Riemannian metric of the surface on which the particles move: (This would essentially just be a local form of Galilean invariance on curved generalizations of the 2D plane, where the kinetic energy is identified with the Laplace-Beltrami operator.).

- Lie Algebra (SL(2,R))

$$[\Lambda^{ab}, \Lambda^{cd}] = \tfrac{1}{2} i \left(\epsilon^{ac} \Lambda^{bd} + \epsilon^{ad} \Lambda^{bc} + \epsilon^{bc} \Lambda^{ad} + \epsilon^{bd} \Lambda^{ac} \right)$$

- quadratic Casimir:

$$[\Lambda^{ab}, C_2] = 0, \quad C_2 = \det \Lambda = \tfrac{1}{2} \epsilon_{ac} \epsilon_{bd} \Lambda^{ab} \Lambda^{cd}$$

- **unitary deformation operator**

$$U(\beta) = \exp i \beta_{ab} \Lambda^{ab} \qquad \beta_{ab} = \beta_{ba}, \quad \text{real}$$

$\det \beta > 0$: pseudo rotation (elliptic)

$\det \beta = 0$: shear (parabolic)

$\det \beta < 0$: squeeze (hyperbolic)

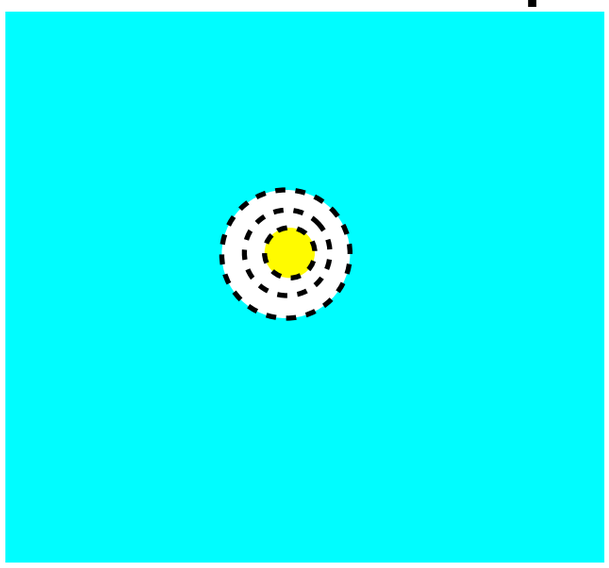
- Laughlin states with different intrinsic metrics are related by transformations

$$|\Psi_L^m(\tilde{g}')\rangle = U(\beta)|\Psi_L^m(\tilde{g})\rangle$$

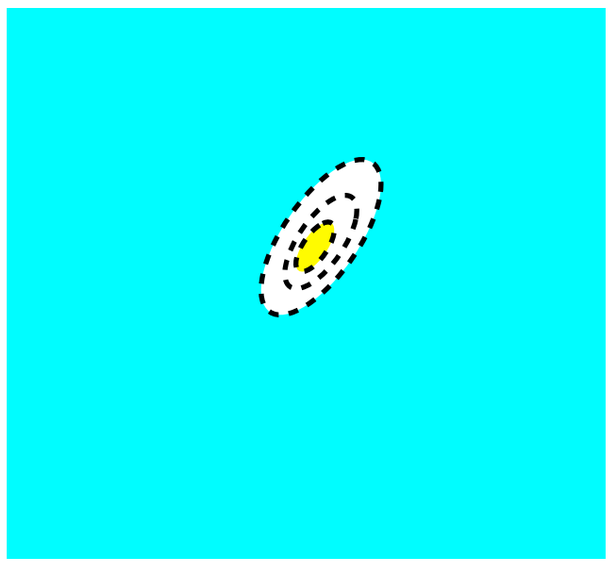
$$\tilde{\boldsymbol{g}}' = \boldsymbol{A}(\beta)\tilde{\boldsymbol{g}}\boldsymbol{A}^T(\beta) \qquad \det \boldsymbol{A}(\beta) = 1 \qquad SL(2,R)$$

- In contrast (in the absence of a boundary), the filled Landau level is not parametrized by a metric, and is left invariant by the action of $U(\beta)$

- The Laughlin state is parametrized by a unimodular metric: what is its physical meaning?



correlation holes
in two states with
different metrics



- In the $\nu = 1/3$ Laughlin state, each electron sits in a correlation hole with an area containing 3 flux quanta. The metric controls the *shap*e of the correlation hole.

- In the $\nu = 1$ filled LL Slater-determinant state, there is <u>no</u> correlation hole (just an exchange hole), and this state does **<u>not</u>** depend on a metric

- Q: If we use the Laughlin state as a variational approximation to a true ground state, what determines the choice of the metric parameter?
- A: it should be chosen to minimize the correlation energy

- The generic Hamiltonian has 2D translation and inversion symmetry, but **does not have** any O(2) rotation symmetry

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad \boldsymbol{R}_i \mapsto \boldsymbol{a} \pm \boldsymbol{R}_i$$

- The uniform incompressible quantum Hall states do not break inversion or translation symmetry, and have no electric polarization.

$$V_n(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi} |f_n(\boldsymbol{q})|^2 \int \frac{d^2\boldsymbol{r}'}{2\pi\ell_B^2} V(\boldsymbol{r}') e^{i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

regular at short distance

Landau level form factor (property of 2D band structure)

Coulomb interaction potential, modified at short distance by finite layer width (property of 3D dielectric tensor)

- If the repulsive short-distance interaction has rotational symmetry with respect to a metric, then the unimodular metric parameter that minimizes the correlation energy will be proportional to that metric.

- The 1/2 (boson) and 1/3 (fermion) Laughlin states with metric $\tilde{g}_{ab}$ are exact zero-energy ground states of a model interaction

$$V_n(\boldsymbol{r}) = (A + Bu(\boldsymbol{r}))e^{-u(\boldsymbol{r})} \quad A, B > 0$$

$$u(\boldsymbol{r}) = \tfrac{1}{2}\tilde{g}_{ab} r^a r^b / \ell_B^2$$

- Let $|\Psi_0\rangle$ be the exact FQH ground state of H

$$\langle\Psi_0|U^{-1}(\beta)HU(\beta)|\Psi_0\rangle \overset{\beta\to 0}{\to} E_0 + \tfrac{1}{2}\Gamma^{abcd}\beta_{ab}\beta_{cd} > E_0$$

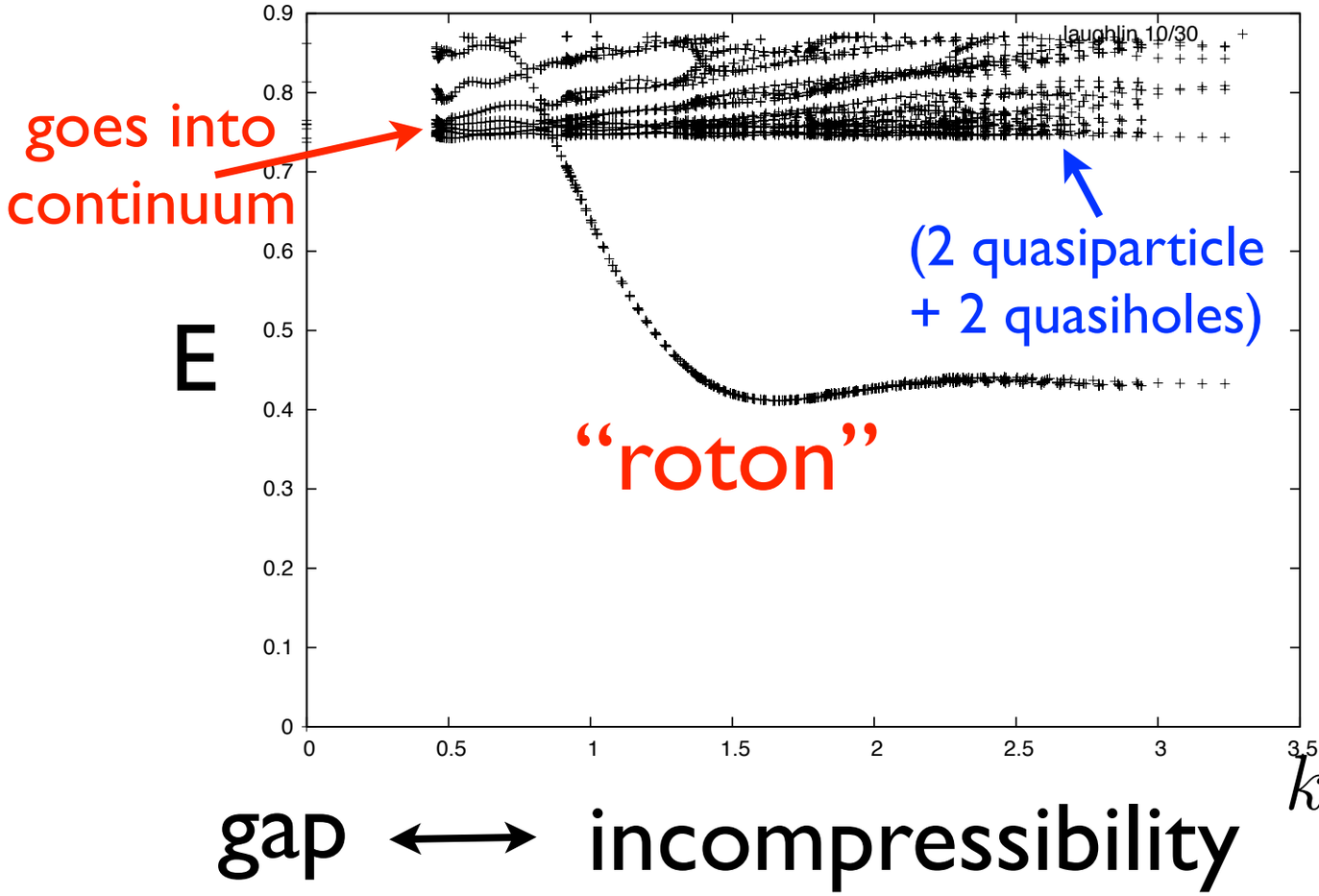
- The rank-4 tensor Γ^{abcd} is a kind of "shear modulus" of the FQH fluid.

- Girvin MacDonald and Platzman found an inequality equivalent as wavevector k → 0 to

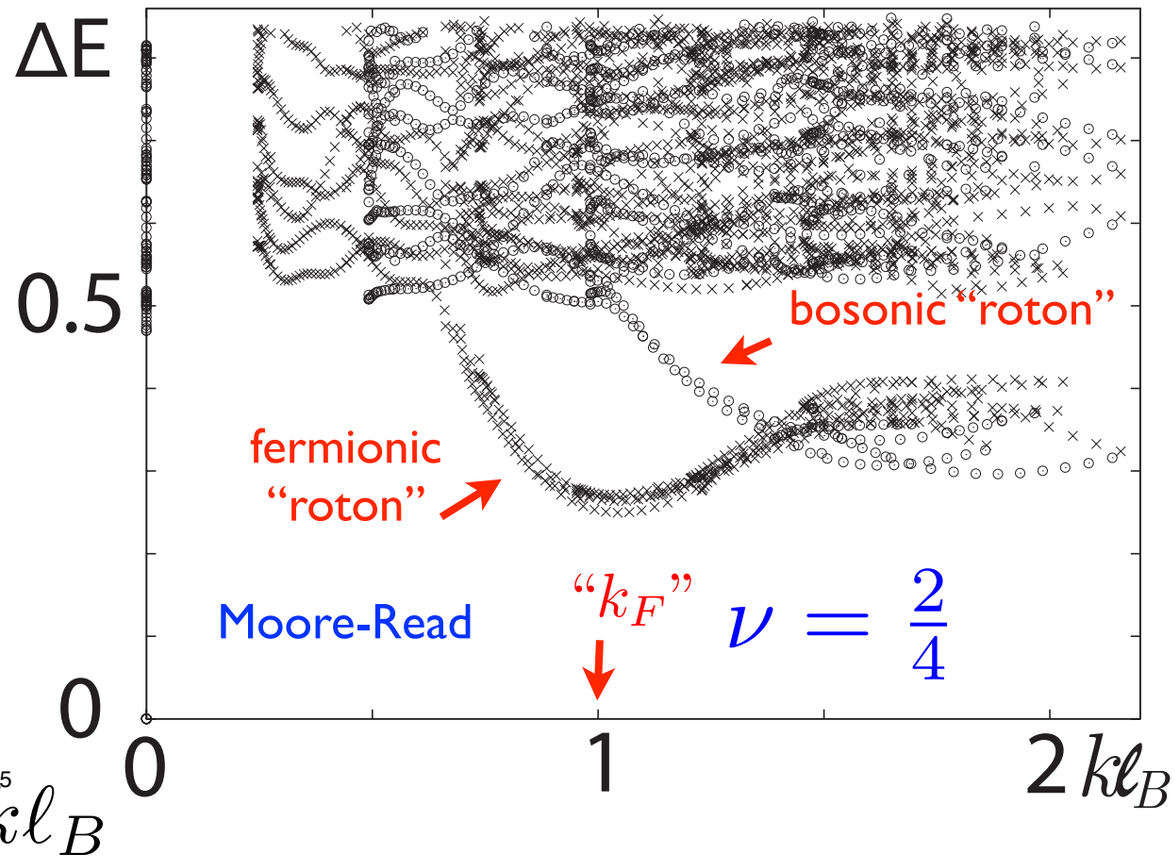
$$S(\boldsymbol{k})\Delta E(\boldsymbol{k}) < \Gamma^{abcd}k_a k_b k_c k d$$

guiding-center structure factor $\propto k^4$

excitation energy constant



gap ⟷ incompressibility



Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

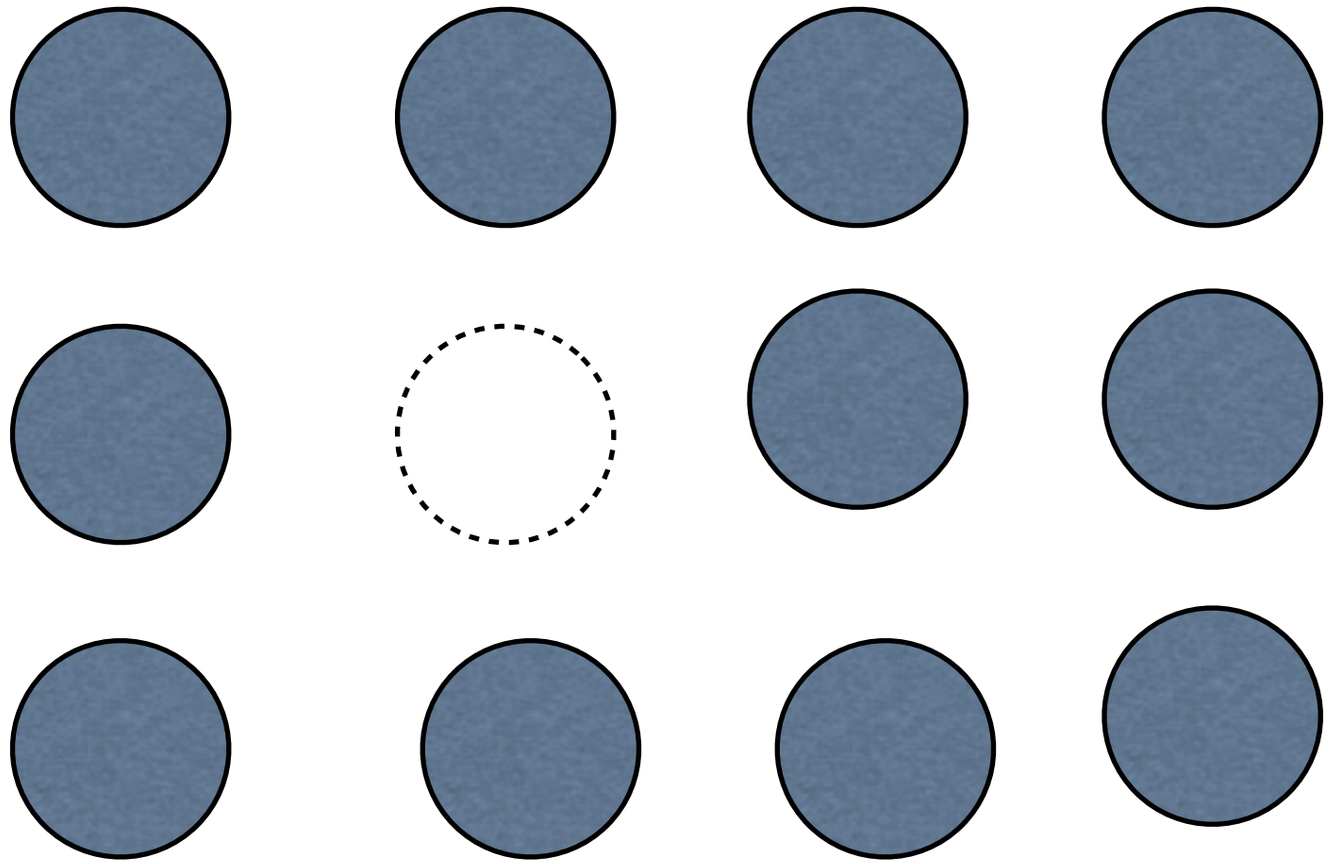
Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar$k of a quasiparticle-quasihole pair is proportional to its **electric dipole moment $\mathbf{p_e}$** $\hbar k_a = \epsilon_{ab} B p_e^b$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: **doesn't transmit pressure!**

(origin of Virasoro algebra in FQHE ?)

- quantum solid
- unit cell is correlation hole
- defines geometry

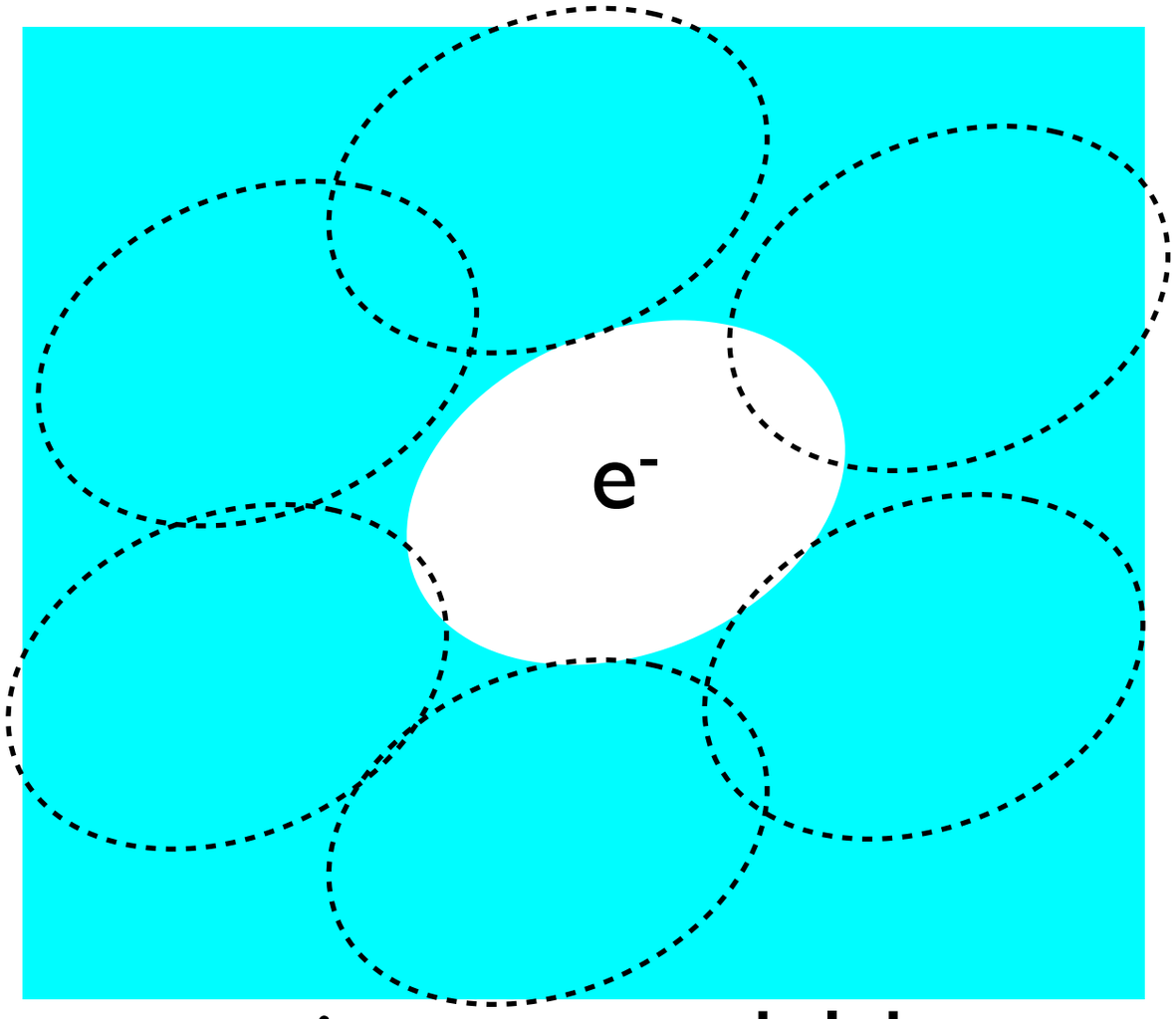


- repulsion of other particles make an attractive potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

but no broken symmetry

- similar story in FQHE:



- continuum model, but similar physics to Hubbard model

- "flux attachment" creates correlation hole

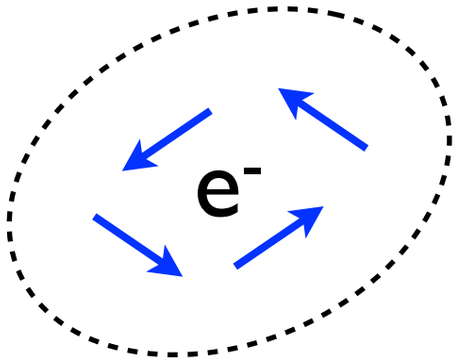
- defines an emergent geometry

- potential well must be strong enough to bind electron

- new physics: Hall viscosity, geometry.....

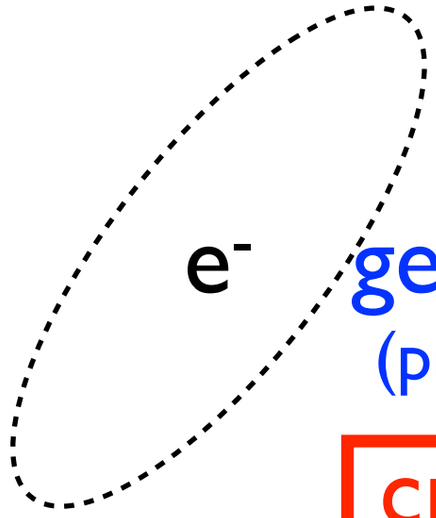
- shape of correlation hole (flux attachment) fluctuates, adapts to environment (electric field gradients)

S



new property:
"spin" couples
to curvature

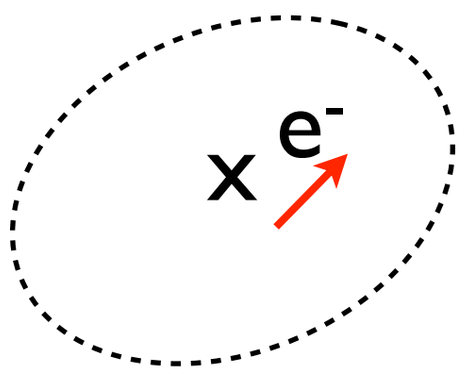
shape=metric



geometric distortion
(preserving inversion symmetry)

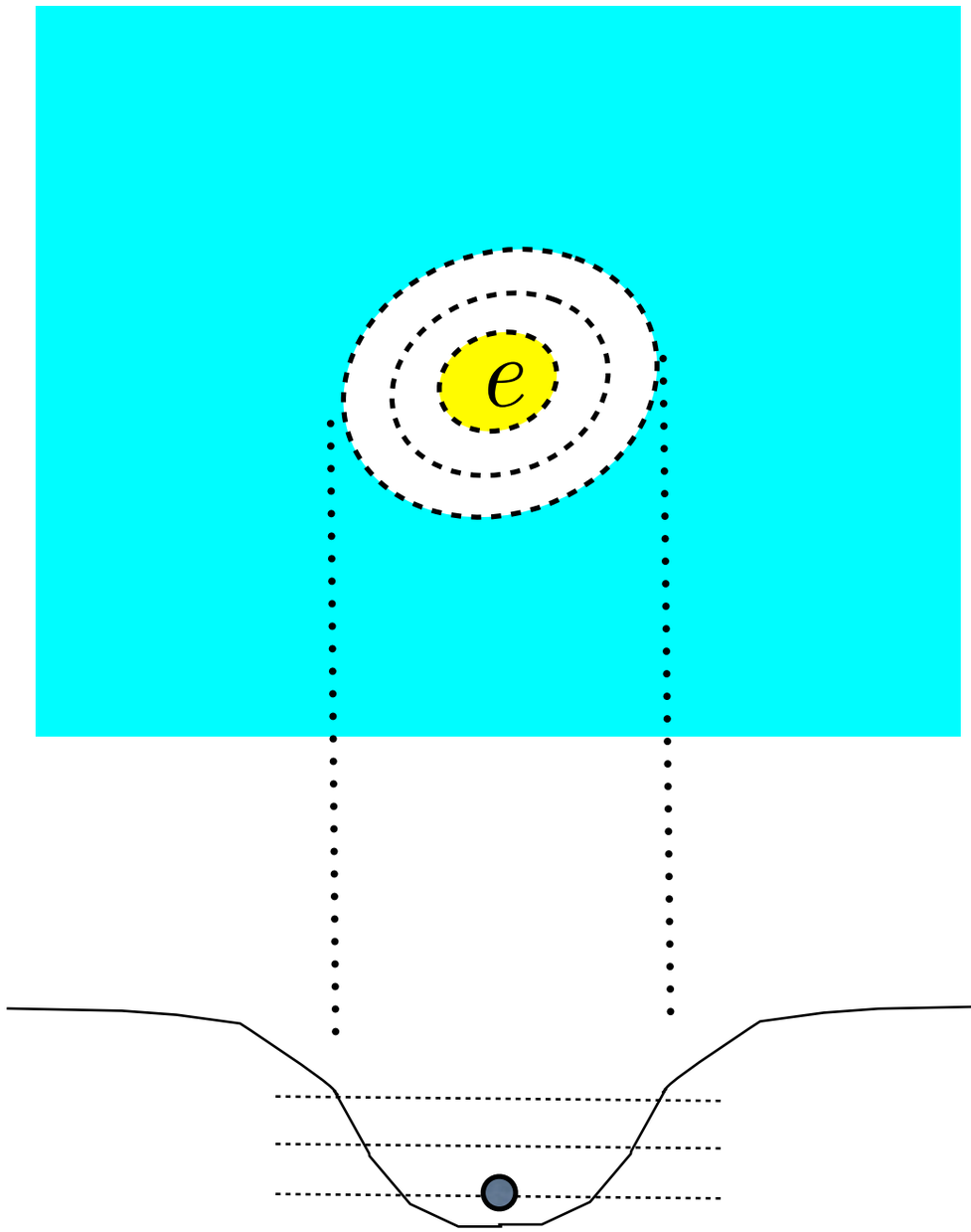
creates "curvature"
of metric

- polarizable, B x electric dipole = momentum, origin of "inertial mass"



electric polarizability

$1/3$ Laughlin state



If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center

It has a "spin"

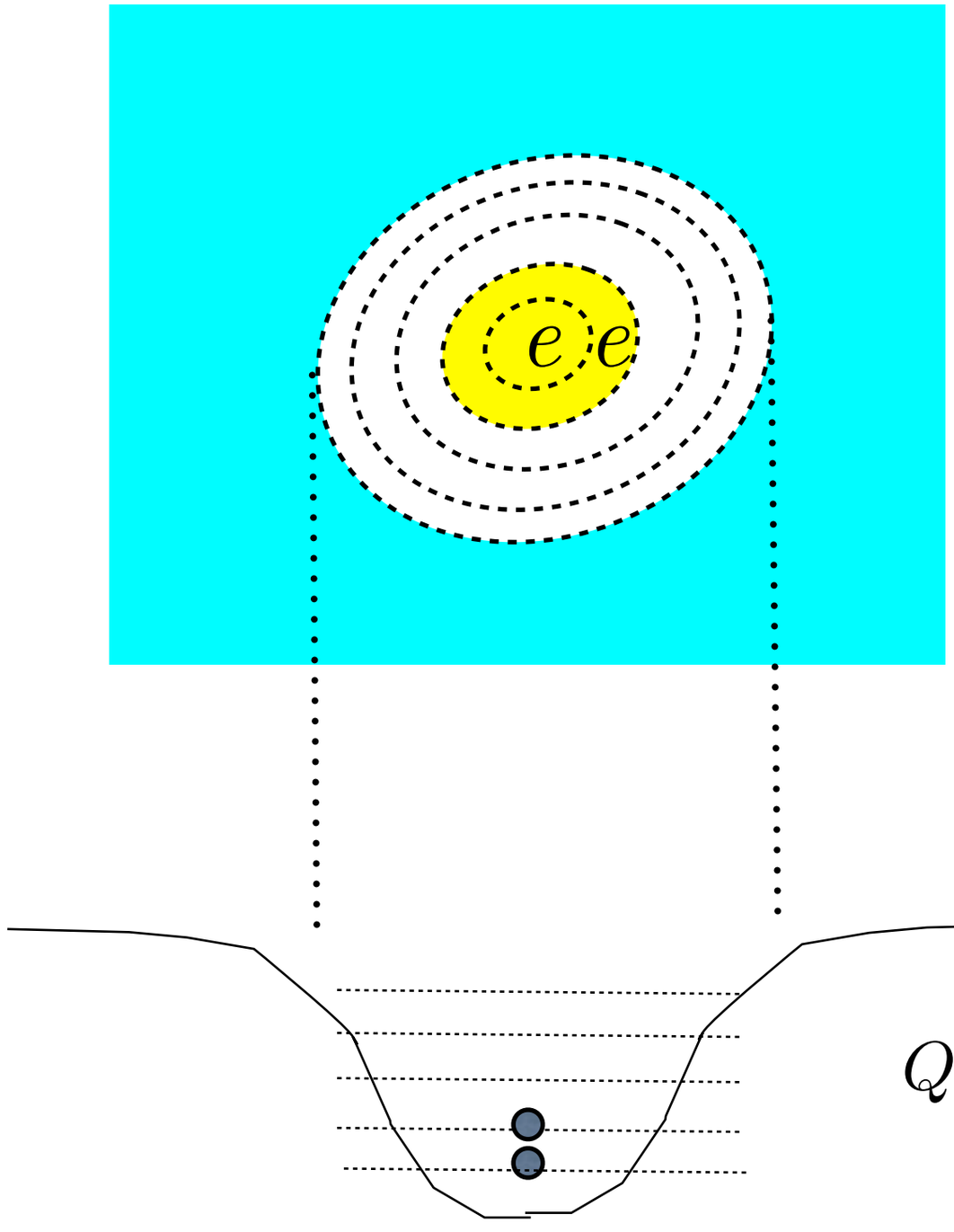
$$\begin{array}{cccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \\ \hline 1 & 0 & 0 & \quad L=\frac{1}{2} \\ \hline \end{array}$$

$$-\quad \begin{array}{|c|c|c|} \hline \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline \end{array} \quad -L=\frac{3}{2}$$

$$s=-1$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



$$
\begin{array}{c|ccccc|c|l}
 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & & & & \\
 & 1 & 1 & 0 & 0 & 0 & \cdots\cdots & L=2 \\
- & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \cdots\cdots & -L=5 \\
\hline
 & & & & & & & s=-3
\end{array}
$$

$$L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b$$

$$Q^{ab} = \int d^2r\, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral composite boson charge distribution

- Now consider the inhomogeneous system

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j) + \sum_i v(\boldsymbol{R}_i)$$

- We must minimize the sum of correlation energy and potential energy
- Now we get a metric with curvature:
- The electron density is tied to

$peB + \hbar S J_g^0$ gaussian curvature density!

"Shift" $= S/p - (n + \frac{1}{2})$

- The shape of the composite boson is determined by minimizing the sum of the correlation energy and the background potential energy.
- If there is no background potential, the metric is flat and the charge density is uniform
- If there is a background potential $g_{ab}(\boldsymbol{r})$ varies with position to give a charge density fluctuation

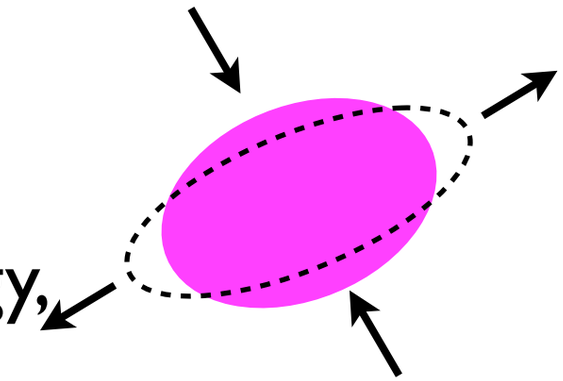
Gaussian curvature of unimodular metric

$$\delta\rho(\boldsymbol{r}) = esK(\boldsymbol{r})$$

$$K(r) = \underbrace{\tfrac{1}{2}\partial_a\partial_b g^{ab}}_{\text{from variation of second moment of charge distribution}} + \underbrace{\tfrac{1}{8}g_{ab}\epsilon_{cd}\epsilon^{ef}\partial_e g^{ac}\partial_f g^{bd}}_{\text{from Berry phase associated with shape change}}$$

"spin"

- The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape



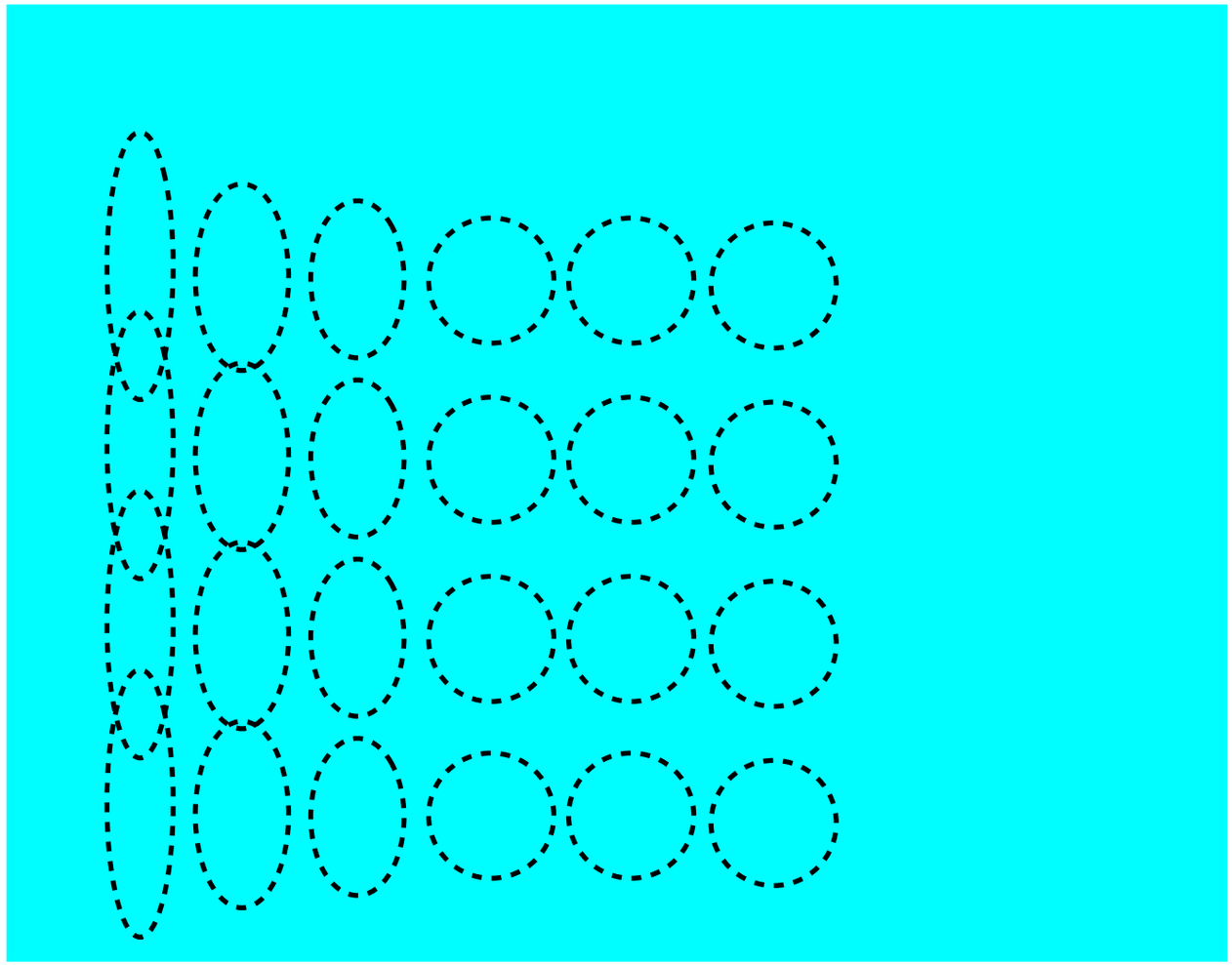
- The zero-point fluctuations of the metric are seen as the O(q^4) behavior of the "guiding-center structure factor" (Girvin et al, (GMP), 1985)

- long-wavelength limit of GMP collective mode is fluctuations of (spatial) metric (analog of "graviton")

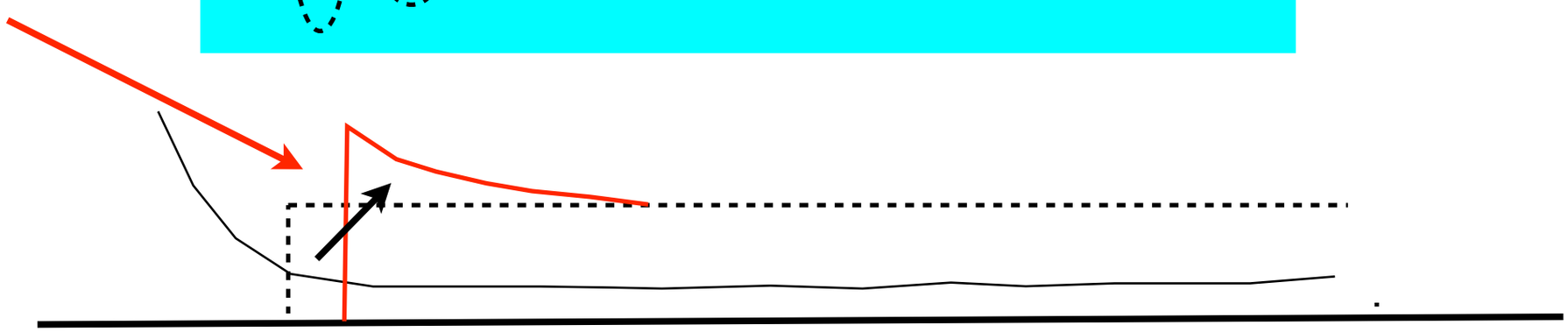
FDMH, Phys. Rev. Lett. **107**, 116801 (2011)

- metric deforms (preserving $\det\ g = 1$)in presence of non-uniform electric field

fluid compressed by Gaussian curvature!



potential near edge

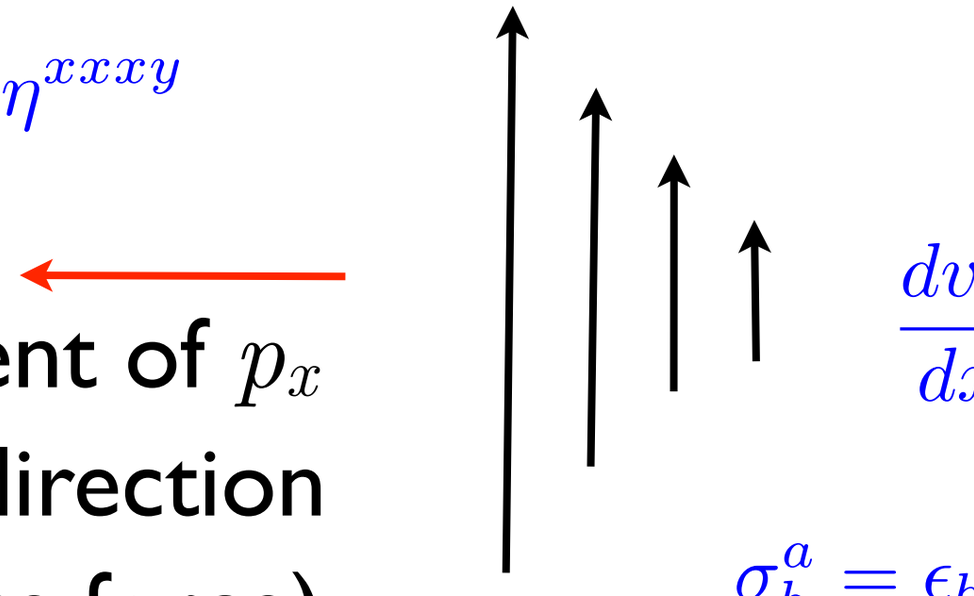


produces a dipole momemt

- Hall viscosity $\eta^{abcd} = \frac{eBs}{4\pi q}\frac{1}{2}\left(g^{ac}\epsilon^{bd} + g^{bd}\epsilon^{ac} + a \leftrightarrow b\right)$

(plus a similar term from the Landau orbit degrees of freedom (Avron et al))

η^{xxxy}



current of p_x in x-direction (stress force)

$\frac{dv^y}{dx}$

$\sigma^a_b = \epsilon_{be}\eta_H^{aecf}\epsilon_{cf}\partial_c v^d$

Hall viscosity determines a dipole moment per unit length at the edge of the fluid

- Total guiding center angular momentum of a fluid disk of N elementary droplets

$$L_{\rm gc} = \frac{1}{2\ell_B^2} g_{ab} \sum_i R^a R_i^b = \frac{1}{2} pq \bar{N}^2 + s_{\rm gc} \bar{N}$$

statistical (conformal) spin

geometric (guiding-center) spin

(dipole at edge)

momentum

$$P_b \doteq B \epsilon_{ab} p^b$$

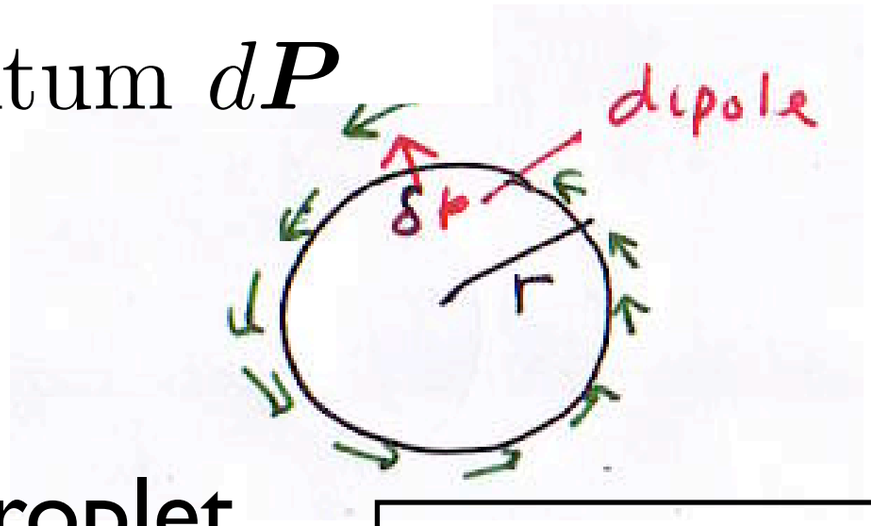
electric dipole

n(k)



orbital occupation

momentum $d\boldsymbol{P}$



circular droplet

The dipole at a segment of the edge has a momentum

$$dP_a = \frac{\hbar}{e\ell_B^2}\epsilon_{ab}dp^b$$

momentum dipole

doesn't contribute to total momentum:

$$\oint dP_a = 0$$

it does contribute an extra term to total angular momentum:

$$\Delta L^z(\boldsymbol{g}) = \hbar \oint \epsilon^{ab} g_{bc} r^c dP_a \neq 0$$