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Nonlinear Dynamics and the Instability of Anti-de Sitter Space.

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NONLINEAR DYNAMICS AND THE (IN)STABILITY OF AdS

Context

Historically, stability considerations and perturbation theory date back to the era of celestial mechanics and the question of stability of the solar system over long time scales. Between 1609 and 1618 Johannes Kepler determined the trajectories of the planets as they revolve around the Sun. Following the work of Copernicus, Kepler placed the Sun at the centre of the universe and based on observations of the famous astronomer of the time Tycho Brache, he succeeded to show that planets move in ellipses around the Sun and at the end of the revolution the planets find themselves back to where they started.

However, this picture of a perfectly stable solar system would be soon challenged. After Isaac Newton developed his theory about gravity, he derived the Keplerian orbits by restricting to the interaction of a planet with the Sun alone. Although this is the leading contribution to the gravitational force exerted to each planet, it is not the only one. Planets attract each other as well. When these perturbations are taken into account they might lead to small effects which accumulate in the course of time destroying in that way the Keplerian orbits.

The study of the stability of the solar system has led to remarkable discoveries in Physics and Mathematics with the most prominent one being perhaps the celebrated Kolmogorov-Arnold-Moser (KAM) theory in which it was rigorously shown that both stable and unstable orbits exist depending on whether the ratio of the unperturbed frequencies is a rational number.

Newton's theory was superseded when Albert Einstein published in 1915 his theory of gravitation, known as General Relativity (GR). According to Einstein, gravity is not a force but rather the manifestation of the geometry of spacetime in which the masses move. Massive objects curve the spacetime and spacetime back-reacts to the masses by dictating them which paths they should follow. Einstein's equations possess three vacuum solutions, namely three different empty spacetimes depending on whether the cosmological constant of the theory is positive (de Sitter), zero (Minkowski) or negative (Anti-de Sitter). The most important question regarding a vacuum state is whether it is stable under small perturbations.

Motivation of research

The stability of the vacuum solutions of GR comes second (perhaps even first) only to the stability of the solar system and has led to one of the greatest developments in mathematical relativity by Christodoulou and Klainerman. Of the three vacuum spacetimes the two where proven to be stable long ago. The stability of the third one (AdS) was not even raised, let alone answered, until very recently.

Anti-de Sitter (AdS) spacetime plays a prominent role in modern Theoretical Physics mainly due to its role in the only concrete example of a gauge/gravity duality, the AdS/CFT correspondence. In this picture, a Quantum Field Theory (QFT) living on the boundary of AdS is equivalent to a String Theory in the AdS background. Despite the great importance of (asymptotically) AdS spacetime(s), the study of its (nonlinear) stability was initiated only very recently, albeit it was earlier conjectured by Dafermos that AdS would be nonlinearly unstable.

Results

Chapter 2 :
In this chapter we presented an alternative and complementary method of studying the problem of the stability of AdS, directly in position space. We derived an approximate/perturbative equation of motion which has a similar scaling symmetry, as the one observed in Fourier space methods (TTF equations). We also showed that the gravitational interaction near the center of the spacetime obeys an exact antisymmetry under time reversal and therefore it is equally likely that the energy be focused or defocused. Finally, we touched on the thermalization process of the boundary field theory and we argued that even if black holes form in the first nonlinear time scale (ϵ^{-2}), it doesn't always represent efficient thermalization of the boundary theory.

Chapter 3 :
Approximating nonlinear dynamics with a truncated perturbative expansion may be accurate for a while, but it in general breaks down at a long time scale that is one over the small expansion parameter (in our case, $t \sim \epsilon^{-2}$). In this chapter we presented cases where such a break down doesn't happen and the perturbation theory is valid up to this time scale, as long as it is applied recursively. There are cases where one can try and *guess* the form of the (regular) solution and then set up a smarter perturbation theory that reproduces this solution. Such is the case of the Two Time Framework (TTF) for example. As we argue in this chapter, if one uses for example a perturbation theory similar to the one used in the second chapter of this thesis, the *regular* solutions of this approximate equation are valid up to $t \sim \epsilon^{-2}$ and not only for $t < \epsilon^{-2}$, as conventional wisdom would suggest. Using these results we then establish the existence of an open set of initial conditions that do not collapse up to this *long time scale*.

Chapter 4 :
An effort to establish collapsing solutions at the vanishing amplitude limit $\epsilon \to 0$ was made in earlier works where solutions of the TTF that develop an *oscillating singularity* were reported. However this singularity is merely a *gauge artefact*.

One can work in a different gauge and not observe this blow up of the derivatives of the phases. In this chapter we showed that these solution are genuine singular solutions and the discrepancy of the results in the two gauges was realized as a diverging redshift between the boundary and the center of the spacetime.

Chapter 5 :
In this chapter we studied the amplitude and the phase dynamics of small perturbations in AdS_4 using the *Two Time Framework* approximation. Our intention was to test the *phase coherent cascade* conjecture of Freivogel and Yang for different initial data. This is done in two ways; either by directly checking the phase coherent ansatz or by studying the resulting power-law for the spectrum of collapsing solutions. Our results suggest that this ansatz works pretty well, however small modifications/improvements might be necessary. We found that the energy spectrum of narrow Gaussian wavepackets scales as $E_n \sim n^{-1}$ and also the phases are coherently aligned ($B_n \sim n$), although some small divergences from this linear behaviour were seen.
We also studied the contentious two-mode equal energy data, and we conjectured that they belong to a new class of solutions that collapse at infinite *slow time* τ, at the vanishing amplitude limit.

Outlook

Stability considerations have led to tremendous discoveries in Theoretical Physics and Mathematics and the stability of AdS could not be an exception to this rule. Although the question has not yet been unequivocally answered, and perhaps there is a long way to go, the studies so far have already unveiled a very rich phenomenology. However there are still many questions to be answered, like what happens for example if we abolish spherical symmetry, or what is the fate of the perturbations at longer time scales.

The problem at hand is not only interesting from the pure mathematical point of view of GR, but can shed light to understanding the thermalization process of the boundary theory via the AdS/CFT correspondence. There have already been very interesting developments in this direction, as well as interesting results in relating quantum revivals (of the boundary QFT) with bouncing geometries in the bulk.